OLD MATH 111 FINAL EXAMS (*Answer not guaranteed to be correct*)

Math 111 Final, Autumn 1995

(1) (15 points) Consider

$$A = \begin{bmatrix} 5 & -4 & -2 & 4 \\ 3 & -2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

1) Find a basis of eigenvectors of A;

2) Use the resulting diagonalization to compute $A^4 - 3A^3 + 2A^2$.

(2) (10 points) Consider polynomials

$$p_1 = x^2 - x, \quad p_2 = 2x^2 - 2x + 1, \quad p_3 = x^2 - 2x.$$

1) Show that p_1, p_2, p_3 is a basis of the space P_2 of polynomials of degree≤ 2;

2) What are the coordinates of $2x^2 + 3x - 1$ with respect to the basis.

(3) (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 3 & 1 & 1 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \\ 1 & 5 & 3 & 5 \end{bmatrix}.$$

Find the rank of A, the dimension of Col$A$, and the dimension of Nul$A$.

(4) (15 points) Consider vector space W spanned by

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 3 \\ -3 \\ 3 \\ 3 \end{bmatrix}, \quad .u_4 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -4 \end{bmatrix}$$

1) Use Gram-Schmidt process to produce an orthogonal basis of W;

2) Find the distance from the vector $(2, 1, 0, 1)$ onto W;

3) Find the distance from the vector $(2, 1, 0, 1)$ onto $W^{\perp}$;

(5) (17 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 5 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}.$$

1) Find an orthogonal basis for Nul$A$;

2) Extend the orthogonal basis of Nul$A$ to an orthogonal basis of $\mathbf{R}^4$.

(6) (15 points)

1) Find an example of 3×3 invertible matrices A and B, such that rank$(A + B) = 1$ (so that $A + B$ is not invertible);

2) Find an example of 3×3 matrix with orthogonal column vectors but *non*orthogonal row vectors;

3) Find an example of 2×2 diagonalizable matrices A and B, such that $A + B$ is not diagonalizable;

(7) (18 points) True or False (no reason needed)

1) If $T : V \to W$ is a linear transformation, and $\{v_1, v_2, v_3\}$ span V, then $\{T(v_1), T(v_2), T(v_3)\}$ span W;

2) If $T : V \to W$ is a linear transformation, and $\{v_1, v_2, v_3\}$ are linearly independent, then $\{T(v_1), T(v_2), T(v_3)\}$ are linearly independent;

3) If $\{v_1, v_2, \cdots, v_n\}$ span V, then $\dim V \le n$;
4) If $\{v_1, v_2, \cdots, v_n\}$ span V, then $\{v_1, v_2, \cdots, v_n, v_{n+1}, \cdots, v_{n+m}\}$ also span V;
5) If $n < \dim V$, then $\{v_1, v_2, \cdots, v_n\}$ are linearly independent;
6) If A is a 11×17 matrix, and the general solution of the $Ax = 0$ has 8 free variables, then $\text{rank}A = 9$.
7) An $n \times n$ matrix can have at most n eigenvalues;
8) If $A \ne 0$, then 0 is not an eigenvalue of A;
9) If A is diagonalizable, then A^T is also diagonalizable;
10) If u is an eigenvector of A and B, then it is an eigenvector of $A + B$;
11) If all eigenvalues of A are 1, then $A = I$;
12) If $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$ are orthogonal sets, then $\{u_1 + u_2, v_1 + v_2, w_1 + w_2\}$ is also an orthogonal set;
13) If $\{v_1, v_2, \cdots, v_n, v_{n+1}, \cdots, v_{n+m}\}$ is orthonormal, then $\{v_1, v_2, \cdots, v_n\}$ is also orthonormal;
14) If the columns of a square matrix A are orthonormal, then the rows of A are also orthonormal;
15) If $\{u_1, u_2\}$ is a basis of W, then $\text{proj}_W v = \text{proj}_{u_1} v + \text{proj}_{u_2} v$;
16) $\text{proj}_W(u + v) = \text{proj}_W u + \text{proj}_W v$;
17) If $u \in V$ is orthogonal to all vectors in V, then $u = 0$;
18) If U is an orthogonal matrix, then U^2 is also orthogonal;

Answer to Math 111 Final, Autumn 1995

(1) The chacteristic equation is $\lambda(\lambda - 1)(\lambda - 2)^2 = 0$. For $\lambda_1 = 0$, we get eigenvector $u_1 = (2, 4, -1, 1)$. For $\lambda_2 = 1$, we get eigenvector $u_2 = (1, 1, 0, 0)$. For $\lambda_3 = 2$, the eigenspace is 2-dimensional with basis $u_3 = (4, 3, 0, 0)$ and $u_4 = (2, 2, 1, 1)$. u_1, u_2, u_3, u_4 is basis of eigenvectors of A.

If $P = [u_1, u_2, u_3, u_4]$, then $\Lambda = P^{-1}AP$ is a diagonal matrix with $0, 1, 2, 2$ as diagonal entries. It follows from $\Lambda^4 - 3\Lambda^3 + 2\Lambda^2 = 0$ that $A^4 - 3A^3 + 2A^2 = 0$.
(2) The coordinates of p_1, p_2, p_3 with respect to the basis $1, x, x^2$ are $u_1 = (0, -1, 1), u_2 = (1, -2, 2), u_3 = (0, -2, 1)$. Since the matrix $[u_1, u_2, u_3]$ is invertible, u_1, u_2, u_3 is a basis of $\mathbf{R}^3$. Thus p_1, p_2, p_3 is a basis of P_2.

If $2x^2 + 3x - 1 = c_1p_1 + c_2p_2 + c_3p_3$, then $c_1 + 2c_2 + c_3 = 2$, $-c_1 - 2c_2 - 2c_3 = 3$, $c_2 = -1$. The solution is $(c_1, c_2, c_3) = (9, -1, -5)$.
(3) A can be reduced by row operations to

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $\text{rank}A = 3$, $\dim\text{Col}A = 3$, and $\dim\text{Nul}A = 4 - 3 = 1$.
(4) By Gram-Schmidt process, we have

$$\begin{array}{l} v_1 = u_1 = (1, 0, 1, 0) \\ v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (2, 2, 2, 0) - \frac{4}{2}(1, 0, 1, 0) = (0, 2, 0, 0) \\ v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 = (3, -3, 0, 3) - \frac{6}{2}(1, 0, 1, 0) - \frac{6}{4}(0, 2, 0, 0) = (0, 0, 0, 3) \end{array}.$$

Since $u_4 \in \text{span}\{v_1, v_2, v_3\}$ (one will get $v_4 = 0$ by Gram-Schmidt process), v_1, v_2, v_3 is an orthogonal basis of W.

We have

$$\text{proj}_W(2, 1, 0, 1) = \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{x \cdot v_3}{v_3 \cdot v_3} v_3 = (1, 1, 1, 1).$$

Thus $\text{dist}((2, 1, 0, 1), W) = \|(2, 1, 0, 1) - \text{proj}_W(2, 1, 0, 1)\| = \sqrt{2}\|$, and $\text{dist}((2, 1, 0, 1), W^\perp) = \|\text{proj}_W(2, 1, 0, 1)\| = 2$.

(5) A is row equivalent to

$$\begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the null space has basis $(-3, 1, 1, 0)$, $(3, -2, 0, 1)$. By Gram-Schmidt process, we get an orthogonal basis $u_1 = (-3, 1, 1, 0), u_2 = (0, -1, 1, 1)$ of Nul$A$.

By the meaning of Nul$A$, the space $(\text{Nul}A)^\perp$ is spanned by the row vectors of A. Thus $(1, 0, 3, -3)$ and $(0, 1, -1, 2)$ is a basis of $(\text{Nul}A)^\perp$. We apply the Gram-Schmidt process in the reversed order to get an orthogonal basis $v_1 = (0, 1, -1, 2)$, $v_2 = (2, 3, 3, 0)$ of $(\text{Nul}A)^\perp$.

u_1, u_2, v_1, v_2 is then an orthogonal basis of $\mathbf{R}^4$.

(6) 1) Take A and B to be diagonal matrices with respectively $1, 1, 1$ and $2, 1, 1$ on the diagonals.

2) If the columns of A are orthogonal, then multiplying some different numbers on the columns of A will still produce orthogonal columns, but will typically produce nonorthogonal rows. For example, $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ has orthogonal columns (and orthogonal rows). But $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ has orthogonal columns but not orthogonal rows. A 3×3 counterexample is given by $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3) $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable. However, $X = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and each summand is diagonalizable (because each has two distinct eigenvalues).

(7) F; F; T; T; F; T; T; F; T; T; F; F; T; T; F; T; T; T.

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(1) (10 points) Consider

$$A = \begin{bmatrix} 1 & 10 & 5 \\ 0 & -4 & 0 \\ 5 & 10 & 1 \end{bmatrix}.$$

1) Find the eigenvalues of A;

2) Find a basis of eigenvectors of A.

(2) (15 points) Consider vector space W spanned by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

1) Show that u_1, u_2, u_3 is a basis of W;

2) Use Gram-Schmidt process to produce an orthogonal basis of W;

3) Find the orthogonal projection of the vector $(12, 0, 0, 0)$ onto W.

(3) (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 8 \end{bmatrix}.$$

1) Find a basis for Col$A$;

2) Find a basis for Nul$A$;

3) Find the rank of A.

(4) (10 points) Find numbers a, b, c, such that

$$U = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix},$$

is an orthogonal matrix. Then find U^{-1}.

(5) (15 points) Consider

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \qquad u = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}.$$

1) Find the distance from u to Nul$A$;
2) Find the distance from u to $(\text{Nul}A)^{\perp}$.

(6) (15 points) Find all the vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

such that

$$\begin{bmatrix} 2 & 1 & 7 & 2 \\ -1 & 2 & -6 & -1 \end{bmatrix} x = 0.$$

(7) (12 points)

1) Find an example of 3×3 *non*invertible matrices A and B, such that $A + B$ is invertible;
2) Find an example of two bases $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$, such that $\{u_1 + w_1, u_2 + w_2, u_3 + w_3\}$ is not a basis;
3) Find an example of invertible 2×2 matrix that is not an orthogonal matrix;
4) Find an example of 2×2 matrices A and B, such that they have the same eigenvalues but A is diagonalizable and B is not diagonalizable;

(8) (13 points) True or False (no reason needed)

1) If v_1, v_2, v_3, v_4 span a 4-dimensional vector space V, then $\{v_1, v_2, v_3, v_4\}$ is a basis of V;
2) If all eigenvalues of A are 0, then $A = 0$;
3) If an invertible matrix A is diagonalizable, then A^{-1} is also diagonalizable;
4) If v is an eigenvector of A, then v is an eigenvector of A^2;
5) If v is an eigenvector of A, then v is an eigenvector of A^T;
6) If u and v are eigenvectors of A, then $u + v$ is also an eigenvector of A;
7) If $\{u, v, w\}$ is an orthogonal basis of $\mathbf{R}^3$, then $\{2u, -3v, 5w\}$ is also an orthogonal basis;
8) If $u \perp v$, $v \perp w$, then $u \perp w$;
9) If $u \in W$ and $u \in W^{\perp}$, then $u = 0$;
10) If the orthogonal projection of y onto W is $\hat{y}$, then the orthogonal projection of $10y$ onto W is $10\hat{y}$;
11) If column vectors of a matrix is orthogonal, then the row vectors of the matrix is also orthogonal;
12) If U and V are orthogonal matrices, then $U + V$ is an orthogonal matrix;
13) If U and V are orthogonal matrices, then UV^{-1} is an orthogonal matrix.

Answer to Math 111 Final, Spring 1995

(1) The chacteristic equation is $(\lambda+4)^2(\lambda-6) = 0$. For $\lambda_1 = -4$, the eigenvectors $x = (x_1, x_2, x_3)$ satisfy $x_1 + 2x_2 + x_3 = 0$. Therefore $u_1 = (-2, 1, 0)$ and $u_2 = (-1, 0, 1)$ is a basis of the eigenspace. For $\lambda_2 = 6$, the eigenvectors x satisfy $x_1 = x_3$, $x_2 = 0$. Therefore $u_3 = (1, 0, 1)$ spans the eigenspace. u_1, u_2, and u_3 is a basis of eigenvectors.

(2) The row operation may reduce the matrix $[u_1, u_2, u_3]$ to $\begin{bmatrix} I_3 \\ 0 \end{bmatrix}$. Since all columns are pivotal, we see that u_1, u_2, u_3 are linearly independent. Since they span W, they form a basis of W.

By Gram-Schmidt process, we have

$$\begin{aligned} v_1 &= u_1 = (1,1,1,1) \\ v_2 &= u_2 - \tfrac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (4,4,4,0) - \tfrac{12}{4}(1,1,1,1) = (1,1,1,-3) \\ v_3 &= u_3 - \tfrac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \tfrac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 = (2,2,0,0) - \tfrac{4}{4}(1,1,1,1) - \tfrac{4}{12}(1,1,1,-3) = \tfrac{2}{3}(1,1,-2,0) \end{aligned}.$$

Thus

$$\mathrm{proj}_W(12,0,0,0) = \frac{x\cdot v_1}{v_1\cdot v_1}v_1 + \frac{x\cdot v_2}{v_2\cdot v_2}v_2 + \frac{x\cdot v_3}{v_3\cdot v_3}v_3 = (6,6,0,0).$$

(3) After row operation, we get

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 8 \end{bmatrix} \longrightarrow B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore the first two columns $(1,0,1,2)$ and $(0,1,0,2)$ form a basis of $\mathrm{Col}A$. Moreover, the solution of $Ax=0$ is given by $x_1=-2x_3$, $x_2=-2x_3$. Therefore $(-2,-2,1)$ is a basis of $\mathrm{Nul}A$. $\mathrm{Rank}A = \dim \mathrm{Col}A = 2$.

(4) U is orthogonal if and only if $U^TU=I$. This means $\frac{1}{3}a+\frac{2}{3}b+\frac{2}{3}c=0$, $\frac{2}{3}a+\frac{1}{3}b-\frac{2}{3}c=0$, $a^2+b^2+c^2=1$. The solution is $(a,b,c)=\pm(\frac{2}{3},-\frac{2}{3},\frac{1}{3})$. If we choose positive sign, then because U is orthogonal,

$$U^{-1} = U^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

(5) $\mathrm{Nul}A$ is spanned by $v_1=(1,1,1)$. So the projection of u onto $\mathrm{Nul}A$ is $\frac{u\cdot v_1}{v_1\cdot v_1}v_1=(2,2,2)$, and the distance from u to $\mathrm{Nul}A$ is $\|u-(2,2,2)\|=2\sqrt{6}$. $(\mathrm{Nul}A)^\perp=\mathrm{Row}A$ is spanned by $v_2=(1,-2,1)$, $v_3=(1,0,-1)$. Since $v_2\perp v_3$, we see that the projection of u onto $(\mathrm{Nul}A)^\perp$ is $\frac{u\cdot v_2}{v_2\cdot v_2}v_2+\frac{u\cdot v_3}{v_3\cdot v_3}v_3=(4,-2,-2)$. Thus the distance from u to $(\mathrm{Nul}A)^\perp$ is $\|u-(4,-2,-2)\|=2\sqrt{3}$.

(6) To be in the span means

$$x = a\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}.$$

Substituting this into the equation we get $12a+4b=0$, $-6a-2b=0$. Thus we get $b=-3a$, and $x=(-2a,a,a,-2a)$. The solutions form a one dimensional subspace spanned by $(-2,1,1,-2)$.

(7)

1) Take A, B to be diagonal, with A having $1,0,0$, and B having $0,1,1$. Then $A+B=I_3$ is invertible.

2) For any basis $\{u_1,u_2,u_3\}$, if we choose $w_i=-u_i$, then $\{u_1+w_1,u_2+w_2,u_3+w_3\}$ consists of 0's and is not a basis;

3) Use any two nonperpendicular vectors of $\mathbf{R}^2$ as the columns;

4) Take $A=I_2$ and B upper triangular with $1,1$ as diagonal.

(8) T; F; T; T;   F; F; T; F;   T; T; F; F; T.

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(1) (17 points) Consider

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

1) Show that A has an orthogonal basis of eigenvectors;

2) Express b as a linear combination of the orthogonal basis found in 1);

3) Use 1) to compute $A^6-6A^4+8A^2+I$.

(2) (17 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \\ 0 & 4 & 0 & 5 & 1 \end{bmatrix}.$$

1) Find a basis for Nul$A$;
2) Find a basis for $(\text{Nul}A)^{\perp}$;
3) Find the rank of A and the dimension of Col$A$.

(3) (15 points) Consider vectors

$$u_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

1) Use Gram-Schmidt process to turn u_1, u_2, u_3 into an orthogonal set;
2) Extend the orthogonal set in 1) to an orthogonal basis of $\mathbf{R}^4$;
3) Find the distance from y to span$\{u_1, u_2, u_3\}$.

(4) (10 points) Consider

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \pi & \sqrt{\pi} & 0 & 0 & 0 \\ e & e^2 & e^3 & 0 & 0 \\ 1 & 10 & 100 & 1000 & 0 \\ 1 & 0.1 & 0.01 & 0.001 & 0.0001 \end{bmatrix}$$

1) Find the eigenvalues of A;
2) Find eigenvalues of $P^{-1}A^{1997}P$;
3) Is A diagonalizable? Explain.

(5) (9 points) Circle the right answer (no reason needed)

1) Suppose the rank of $A_{5\times 8}$ is 3. Then the dimension of $(\text{Nul}A^T)^{\perp}$ is

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8.$$

2) Suppose x is the orthogonal projection of y to $W^{\perp}$. Then the projection of $-y$ to W is

$$y + x, \quad y - x, \quad -y - x, \quad -y + x$$

3) Suppose the eigenvalues of $A^2 - 3A + 2$ are 0 and 2. Then the possible eigenvalues of A^2 are

$$\{0,1,2,3\}, \quad \{0,1,4,9\}, \quad \{0,2\}, \quad \{0,4\}, \quad \{0,1,2,3,4,9\}.$$

(6) (12 points)

1) Find an example of 3×3 matrices A and B, such that A and B have the same eigenvectors but not the same eigenvalues;

2) Find an example of 3×3 matrices A and B, such that A and B have the same eigenvalues but not the same eigenvectors;

3) Find an example of 2×2 matrices A and B, such that A and B are not diagonalizable but $A + B$ is diagonalizable;

4) Find an example of 2×2 matrix A such that A is not diagonalizable, but A^2 is diagonalizable.

(7) (20 points) True or False (no reason needed)

1) If columns of $A_{5\times 3}$ are linearly independent, then rows of A span $\mathbf{R}^3$;
2) If $A_{m\times n}x = b$ always has solution, then rank$A = m$;
3) If $A_{m\times n}x = 0$ has only trivial solution, then rank$A = m$;
4) If columns of a square matrix A are orthogonal, then the rows of A are orthogonal;
5) If columns of a square matrix A are orthonormal, then the rows of A are orthonormal;
6) If all eigenvalues of A are 1, then $A = I$;
7) If A and B are diagonalizable, then AB is also diagonalizable;
8) If v is an eigenvector of A, then v is an eigenvector of $2A$;
9) If v is an eigenvector of A and B, then v is an eigenvector of AB;
10) If λ is an eigenvalue of A and B, then λ is an eigenvalue of $A + B$;

11) If λ is an eigenvalue of A and B, then λ is an eigenvalue of AB;
12) If $A_{n\times n}$ has fewer than n distinct eigenvalues, then A is not diagonalizable;
13) Distinct eigenvectors are linearly independent;
14) If $u \perp w$, $v \perp w$, then $u + v \perp w$;
15) If $u \perp u$, then $u = 0$;
16) If r is any number, then $\|ru - rv\| = r\|u - v\|$;
17) If $\|u - v\|^2 = \|u\|^2 + \|v\|^2$, then $u \perp v$;
18) If $u \perp v$, then $\|u - v\|^2 = \|u\|^2 + \|v\|^2$;
19) If U is and orthogonal matrix, then U^T is an orthogonal matrix;
20) If U is and orthogonal matrix, then U^{-1} is an orthogonal matrix.

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(1) $\det(A - \lambda I) = \lambda(\lambda - 2)(\lambda^2 - 2)$. Therefore we obtain $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = \sqrt{2}, \lambda_4 = -\sqrt{2}$ and the corresponding eigenvectors (your answer may differ by multiplying a number)

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} \sqrt{2}-1 \\ -1 \\ 1 \\ -\sqrt{2}+1 \end{bmatrix}, \quad u_4 = \begin{bmatrix} -\sqrt{2}-1 \\ -1 \\ 1 \\ \sqrt{2}+1 \end{bmatrix}.$$

Then we may easily verify that $u_i \cdot u_j = 0$ for $i \neq j$. Therefore u_1, u_2, u_3, u_4 is an orthogonal basis of eigenvectors of A.

The expression of b in terms of the orthogonal basis is

$$b = \frac{b \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots = \frac{1}{2}(u_1 + u_2 + u_3 + u_4).$$

From 1) we have $A = U^{-1}DU$ for diagonal matrix D with $0, 2, \sqrt{2}$ and $-\sqrt{2}$ on the diagonal. Therefore $A^6 - 6A^4 + 8A^2 + I = U^{-1}(D^6 - 6D^4 + 8D^2 + I)U = U^{-1}IU = I$.

(2) A can be row reduced to

$$B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the general solution for $Ax = 0$ is $a(2, -5, 1, 4, 0) + b(-2, -1, 1, 0, 4)$, and the vectors $(2, -5, 1, 4, 0)$, $(-2, -1, 1, 0, 4)$ form a basis of $\text{Nul}A$.

Since $(\text{Nul}A)^\perp = \text{Row}A$, the first three rows of B form a basis of $(\text{Nul}A)^\perp$.

$\text{rank}A = \dim \text{Col}A = \dim \text{Row}A = 3$.

(3) The Gram-Schmidt process produces the orthogonal set

$$v_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix}.$$

The extra vector in 2) is obtained by solving $u_1 \cdot x = u_2 \cdot x = u_3 \cdot x = 0$, which gives $v_4 = (1, 1, -2, 3)$. The distance from y to $\text{span}\{u_1, u_2, u_3\}$ is the length of the projection of y in v_4 direction, which is

$$\text{dist} = \|\frac{y \cdot v_4}{v_4 \cdot v_4} v_4\| = \frac{|y \cdot v_4|}{\|v_4\|} = \sqrt{\frac{3}{5}}.$$

(4) The eigenvalues of A are $1, 2, 3, 4, 5$. Since A has five distinct eigenvalues, A is diagonalizable. The eigenvalues of $P^{-1}A^{1997}P$ are the same as the eigenvalues of A^{1997}, which are $1^{1997} = 1, 2^{1997}, 3^{1997}, 4^{1997}, 5^{1997}$.

(5) 1) $\dim(\text{Nul}A^T)^\perp = \dim \text{Col}A = \text{rank}A = 3$;

2) The projection of $-y$ to $W^\perp$ is $-x$. Therefore the projection of $-y$ to W is $(-y)-(-x) = x - y$;

3) $\lambda^2 - 3\lambda + 2 = 0, 2 \Longrightarrow \lambda = 0, 1, 2, 3 \Longrightarrow \lambda^2 = 0, 1, 4, 9$.

(6) 1) If a, b, c are distinct, then the eigenvector of $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ are e_1, e_2, e_3, and eigenvalues are a, b, c. We may choose A to be diagonal with $a = 0, b = 1, c = 2$, and B with $a = 2, b = 0, c = 1$.

2) If $B = P^{-1}AP$, then A and B have the same eigenvalues but the eigenvectors are transformed by P. From this it is easy to construct example.

3) Choose A not diagonalizable. Then $B = -A$ is not diagonalizable. But $0 = A + B$ is diagonal.

4) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(7) T; T; F; F; T;   F; F; T; T; F;   F; F; F; T; T;   F; T; T; T; T.

Math 111 Final, Autumn 1997

(1) (10 points) Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

1) Find the rank of A;

2) Find the dimensions of the row space $R(A^t)$, the column space $R(A)$, the null $N(A)$ of A, and the null $N(A^t)$ of A^t.

(2) (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 4 & -3 & -2 \\ 1 & -1 & 2 & 3 \end{bmatrix}.$$

1) Find an orthogonal basis of the column space $R(A)$;

2) Extend the orthogonal basis of $R(A)$ to an orthogonal basis of the whole Euclidean space;

3) Find an orthogonal basis of the orthogonal complement of the row space $R(A)$;

4) Find the distance from $(1, 1, 1, 1)$ to the row space $R(A)$.

(3) (15 points) Consider

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$

1) Find eigenvalues and eigenvectors of A;

2) What are the geometric and algebraic multiplicities of the eigenvalues;

3) Is A diagonalizable? If not, give reason. If yes, find invertible S and diagonal D, such that $A = SDS^{-1}$.

(4) (10 points) Consider the matrix

$$A = \begin{bmatrix} -7 & 24 \\ -2 & 7 \end{bmatrix}.$$

1) Diagonalize A;

2) Use the diagonalization of A to compute A^{1997}. Do not try to directly multiply A 1997 times.

(5) (15 points) Consider the space P^4 with inner product $< p, q >= \int_{-1}^{1} p(x)q(x)dx$. Consider the subspace S spanned by $1 + x^2, (1 - x)(1 + x^2), x(1 + x^2)$.

1) Find an orthogonal basis of S;

2) Find the orthogonal projection of 1 on S;

3) Find the orthogonal projection of 7 on $S^\perp$.

(6) (12 points)

1) Find an example of two linearly dependent sets $\{v_1, v_2, v_3\}$, $\{w_1, w_2, w_3\}$, such that $\{v_1 + w_1, v_2 + w_2, v_3 + w_3\}$ is linearly independent;

2) Find an example of 3×3 matrices A and B, such that rank of $A = 1$, nullity of $B = 1$, and rank of $(A + B) = 3$.

3) Find an example of 2×2 matrices A and B, such that A and B are not diagonalizable but AB is diagonalizable;

4) Find an example of 2×2 matrices A and B, such that A and B are diagonalizable but AB is not diagonalizable.

(7) (18 points) True or False (no reason needed)

1) Rank of $A_{7\times5}$ is 5 $\Longrightarrow$ Rows of A linearly independent;

2) Rank of $A_{7\times5}$ is 5 $\Longrightarrow$ Columns of A linearly independent;

3) Rank of $A_{7\times5}$ is 5 $\Longrightarrow \dim N(A) = 0$;

4) Rank of $A_{7\times5}$ is 5 $\Longrightarrow \dim N(A^t) = 0$;

5) Rank of $A_{7\times5}$ is 5 $\Longrightarrow$ Solution of $Ax = 0$ is unique;

6) Rank of $A_{7\times5}$ is 5 $\Longrightarrow Ax = b$ has solution for all b;

7) $\{u_1, \cdots, u_n\}$ is an orthogonal basis with respect to $<,> \Longrightarrow \{u_1, \cdots, u_n\}$ is an orthogonal basis with respect to $2 <,>$;

8) $\{u_1, \cdots, u_n\}$ is an orthonormal basis with respect to $<,> \Longrightarrow \{u_1, \cdots, u_n\}$ is an orthonormal basis with respect to $2 <,>$;

9) $u \perp w$, $v \perp w \Longrightarrow 2u - 3v \perp w$;

10) $u \perp v$ and L is a linear transformation $\Longrightarrow L(u) \perp L(v)$;

11) The orthogonal projection of u and v on S are x and $y \Longrightarrow$ The orthogonal projection of $u + v$ on S is $x + y$;

12) The orthogonal projection of u on S is $x \Longrightarrow$ The orthogonal projection of $2u$ on $S^\perp$ is $2u - 2x$;

13) λ is an eigenvalue of $A \Longrightarrow \lambda$ is an eigenvalue of A^t;

14) v is an eigenvector of $A \Longrightarrow v$ is an eigenvector of A^t;

15) A is diagonalizable $\Longrightarrow A^2$ is diagonalizable;

16) λ is an eigenvalue of A and μ is an eigenvalue of $B \Longrightarrow \lambda\mu$ is an eigenvalue of AB;

17) $A \neq I \Longrightarrow 1$ is not an eigenvalue of A;

18) If A and B have the same eigenvalues, then A is diagonalizable $\Longrightarrow B$ is diagonalizable.

Answer to Math 111 Final, Autumn 1997

(1) After row operation, A becomes

$$\begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the rank of A is 3. The dimensions of the row space and the column space are 3, because they are the same as the rank. The dimension of the null of A is $5 - 3 = 2$, and the dimension of the null of A^t is $4 - 3 = 1$.

(2) After row operation, A becomes

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis of $R(A)$ is the first two columns $(1, 1, 1)$, $(0, 4, -1)$. By Gram-Schmidt, we get orthogonal basis $v_1 = (1, 1, 1)$, $v_2 = (1, -3, 2)$ of $R(A)$. We need one more vector for an orthogonal basis of $\mathbf{R}^3$. Solving $< v_1, v >= 0$, $< v_2, v >= 0$ gives $v = c(-5, 1, 4)$. Thus v_1, v_2, $v_3 = (-5, 1, 4)$ is an orthogonal basis of $\mathbf{R}^3$.

The orthogonal complement of $R(A^t)$ is $N(A)$. From the row reduction of A we get a basis $(-1, 1, 1, 0)$, $(-2, 1, 0, 1)$. By Gram-Schmidt, we get orthogonal basis $w_1 = (-1, 1, 1, 0)$, $w_2 = (1, 0, 1, -1)$ of $N(A) = R(A^t)^\perp$.

The orthogonal projection of $(1,1,1,1)$ onto $R(A^t)^\perp$ is $1/3w_1+1/3w_2 = 1/3(0,1,2,-1)$. The distance from $(1,1,1,1)$ to $R(A^t)$ is $\|1/3(0,1,2,-1)\| = \sqrt{2/3}$.
(3) The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$. The algebraic multiplicities for both eigenvalues are 2.

For $\lambda_1 = 1$, the eigenspace has basis $v_1 = (0,0,0,1)$. Hence the geometric multiplicity for λ_1 is 1. For $\lambda_2 = 2$, the eigenspace has basis $v_2 = (0,0,1,2)$. Hence the geometric multiplicity for λ_2 is also 1.

A is not diagonalizable because we have only two linearly independent eigenvectors. Four such vectors are needed for diagonalization.
(4) The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = -1$. The eigenspaces have bases $(3,1)$, $(4,1)$. Thus

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1}.$$

and

$$A^{1997} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{1997} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1} = A = \begin{bmatrix} -7 & 24 \\ -2 & 7 \end{bmatrix}.$$

(5) A basis of S is $1+x^2, (1-x)(1+x^2)$. By applying Gram-Schmidt to this, we obtain orthogonal basis $1+x^2, x(1+x^2)$ of S. By the usual formula, the orthogonal projection of 1 onto S is $5/7(1+x^2)$. Then the orthogonal projection of 7 onto S is $7(5/7(1+x^2)) = 5+5x^2$, and the orthogonal projection of 7 onto $S^\perp$ is $7-(5+5x^2) = 2-5x^2$.
(6)

1) Let $\{u_1, u_2, u_3\}$ be independent. Take $\{v_1, v_2, v_3\} = \{u_1, u_2, 0\}$ and $\{w_1, w_2, w_3\} = \{0, 0, u_3\}$;

2) A is diagonal with $1,0,0$ on diagonal. B is diagonal with $0,1,1$ on diagonal. The $A+B = I$ has rank 3;

3) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are not diagonalizable. $AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is diagonalizable (two distinct eigenvalues);

4) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$ are diagonalizable (two distinct eigenvalues). $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

(7) F, T, T, F, T, F; T, F, T, F, T, T; T, F, T, F, F, F.

Math 111 Final, Spring 1998

(1) (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}.$$

1) Find a basis for Col$A$;
2) Find a basis for Nul$A$;
3) Find the rank of A and dimensions of Col$A$, Row$A$, Nul$A$, NulA^T;
4) Show that first, third, and the fifth columns of A linearly independent;
5) Is it possible to find four columns of A that are linearly independent? Explain.

(2) (20 points) Consider

$$A = \begin{bmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

1) Find an orthogonal basis of eigenvectors for A;
2) Find the orthogonal projections of b onto the eigenspaces;
3) Find the distances from b to the eigenspaces;

4) Compute $A^6 - 72A^4 + 1296A^2$ (Note: $1296 = 36^2$);

5) Find the quadratic form $Q(x_1, x_2, x_3)$ corresponding to A and use an orthogonal change of variable to transform Q into one with no cross-product term.

(3) (15 points) Consider polynomials $p_1(t) = -1 + 2t + t^2$, $p_2(t) = 1 + 2t + t^3$, $p_3(t) = -2 + t^2 - t^3$, $q(t) = a + 2t - t^2 + (a-1)t^3$.

1) For what a is q in the span of p_1, p_2, p_3;

2) Find a basis of the span of p_1, p_2, p_3;

3) For the a you find in 1), find the coordinates of q relative to the basis you find in 2);

4) Extend the basis in 2) to a basis of P_3, the space of polynomials of degree ≤ 3.

(4) (15 points) Consider the subspace V of $\mathbf{R}^3$ given by $x_1 - x_2 + 2x_3 = 0$.

1) Find an orthogonal basis of V;

2) Find the matrix P for the orthogonal projection onto V;

3) Find a diagonalization of P.

(5) (10 points) For which a, the matrix $A = \begin{bmatrix} 1 & -a \\ a & 3 \end{bmatrix}$ is diagonalizable. For which a, the matrix A is not diagonalizable. Please provide full explanation.

(6) (20 points) True or False (no reason needed)

1) If all eigenvalues of A are nonzero, then A is invertible;

2) If all eigenvalues of A are zero, then $A = O$;

3) If A is diagonalizable, then A^3 is diagonalizable;

4) If A^3 is diagonalizable, then A is diagonalizable;

5) If vectors $v_1, v_2, \cdots, v_k$ are orthogonal, then they are linearly independent;

6) If vectors $v_1, v_2, \cdots, v_k$ are orthonormal, then they are linearly independent;

7) $\text{proj}_W(2u + 3v) = 2\text{proj}_W u + 3\text{proj}_W v$;

8) $\text{proj}_{\text{span}(u_1,u_2)} v = \text{proj}_{u_1} v + \text{proj}_{u_2} v$

9) If u is an eigenvector of A and B, then u is an eigenvector of AB;

10) If u and v are eigenvectors of A and B, then $u + v$ is an eigenvector of $A + B$;

11) If u is an eigenvector of A, then u is an eigenvector of A^T;

12) If λ is an eigenvalue of A, then λ is an eigenvalue of A^T;

13) If columns of an $n \times n$ matrix U is an orthonormal basis of $\mathbf{R}^n$, then $U^T U$ is diagonal;

14) If for an $n \times n$ matrix U, $U^T U$ is diagonal, then the columns of U an orthonormal basis of $\mathbf{R}^n$;

15) If $A_{3\times 5} x = b$ has solution for all b, then $\text{rank}A = 3$;

16) If $A_{3\times 5} x = b$ has solution for all b, then $\text{rank}A = 5$;

17) If $A_{3\times 5} x = b$ has solution for all b, then $A^T x = 0$ has only trivial solution;

18) Rank of $A_{3\times 5}$ is 3 implies columns of A are linearly independent;

19) Rank of $A_{3\times 5}$ is 3 implies rows of A are linearly independent;

20) Rank of $A_{3\times 5}$ is 3 implies $Ax = 0$ has only trivial solution.

Answer to Math 111 Final, Spring 1998

(1) A can be row reduced to

$$B = \begin{bmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the first, second, and the fifth columns of A forms a basis of Col$A$. The general solution for $Ax = 0$ is $x_3(-3, -1, 1, 0, 0) + x_4(-7, -3, 0, 1, 0)$, and the vectors $(-3, -1, 1, 0, 0)$, $(-7, -3, 0, 1, 0)$ form a basis of Nul$A$.

rank$A = 3$, dimCol$A = 3$, dimRow$A = 3$, dimNul$A = 2$, dimNul$A^T = 1$.

The same row operations that takes A to B carries the $(1, 3, 5)$-columns of A to the $(1, 3, 5)$-columns of B. Since in the $(1, 3, 5)$-column matrix of B, all three columns are pivots, we see that the $(1, 3, 5)$-columns of A are linearly independent.

The maximal number of linearly independent columns of A is rank$A = 3$. Therefore any four columns of A must be linearly dependent.

(2) $\det(A-\lambda I) = -\lambda(6+\lambda)^2$. For the eigenvalue $\lambda = 0$, the eigenspace E_0 has basis $v_1 = (-1,-2,1)$. For the eigenvalue $\lambda = -6$, the eigenspace E_{-6} has basis $v_2 = (1,0,1)$, $v_3 = (-2,1,0)$. Applying Gram-Schmidt to v_2, v_3, we get an orthogonal basis $v_2 = (1,0,1)$, $u_3 = (-1,1,1)$ for E_{-6}. Thus v_1, v_2, u_3 is an orthogonal basis of eigenvectors of A. Dividing length, this leads to

$$A = UDU^{-1}, \qquad U = \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix},$$

where U is an orthogonal matrix.

From the usual formula, the orthogonal projections $\text{proj}_{E_0} b = (-1,-2,1)$, $\text{proj}_{E_{-6}} b = (1,1,3)$. Then $dist(b, E_0) = ||(1,1,3)\| = \sqrt{11}$, $dist(b, E_{-6}) = ||(-1,-2,1)\| = \sqrt{6}$.

Since $D^6 + 72D^4 + 1296D^2 = O$, we have $A^6 + 72A^4 + 1296A^2 = O$.

The quadratic form corresponding to A is $Q(x_1,x_2,x_3) = -5x_1^2 - 2x_2^2 - 5x_3^2 + 4x_1x_2 - 2x_1x_3 - 4x_2x_3$. With the orthogonal change of variable $x = Uy$, we get $Q = -6y_2^2 - 6y_3^2$.

(3) The problem is turns into a problem in $\mathbf{R}^4$ with vectors $[p_1] = (-1,2,1,0)$, $[p_2] = (1,2,0,1)$, $[p_3] = (-2,0,1,-1)$, and $[q] = (a,2,-1,a-1)$. Row operation on the matrix $([p_1],[p_2],[p_3],[q])$ gives

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & a-3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The answer for 1) is then $a = 3$. We also find p_1, p_2 is a basis of the span of p_1, p_2, p_3. Moreover, by deleting the third column we find the solution $x_1 = -1$ and $x_2 = 2$ in case $a = 3$. Therefore $q(t) = 3 + 2t - t^2 + 2t^3$ has coordinate $(-1,2)$ relative to the basis p_1, p_2.

To extend the basis to a basis of P_3, we note that $[p_1]$, $[p_2]$, e_1, e_2 for a basis of $\mathbf{R}^4$, therefore p_1, p_2, 1, t form a basis of P_3.

(4) A basis of V is given by $(1,1,0)$, $(-2,0,1)$. Applying Gram-Schmidt to the basis, we get an orthogonal basis $(1,1,0)$, $(-1,1,1)$.

The orthogonal projection of $x = (x_1,x_2,x_3)$ onto V is $\frac{x_1+x_2}{2}(1,1,0) + \frac{-x_1+x_2+x_3}{3}(-1,1,1) = \frac{1}{6}(5x_1 + x_2 - 2x_3, x_1 + 5x_2 + 2x_3, -2x_1 + 2x_2 + 2x_3)$. The matrix of the projection is then

$$P = \frac{1}{6}\begin{bmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

P has eigenvectors $(1,1,0)$, $(-1,1,1)$, with eigenvalue 1, and eigenvector $(1,-1,2)$, with eigenvalue 0. Therefore

$$P = UDU^{-1}, \qquad U = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(5) $\det(A-\lambda I) = \lambda^2 - 4\lambda + 3 + a^2$. Therefore A has two distinct eigenvalues $\Leftrightarrow a \neq \pm 1$. In this case, A is diagonalizable.

In case $a = 1$, we have $\lambda = 2$ and $A - \lambda I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ and can find only one eigenvector. Therefore A is not diagonalizable. Similarly, if $a = -1$, then $\lambda = 2$ and $A - \lambda I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Again we can find only one eigenvector. Therefore A is still not diagonalizable.

(6) T; F; T; F; F; T; T; F; T; F; F; T; T; F; T; F; T; F; T; F.

Math 111 Final, Autumn 1999, section 1

(1) (10 points) Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \qquad u = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}.$$

1) Find eigenvalues and eigenvectors of A;
2) Can you find invertible P and diagonal D, such that $A = PDP^{-1}$? Explain;
3) Find general formula for $A^n u$.

(2) (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 \\ -2 & 1 & 1 & 0 \\ -1 & 1 & -2 & 1 \\ 0 & 1 & -5 & 2 \\ 3 & -1 & -4 & 1 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} 1 \\ -1 \\ a-1 \\ 1 \\ a+1 \end{bmatrix}$$

1) Find an orthogonal bases of Col$A$;
2) Find the orthogonal projection of v on Col$A$;
3) Determine a for which w is in Col$A$;
4) Find the orthogonal projection of $-2v$ on NulA^T;
5) What is the maximal number of linearly independent rows of A? Explain;
6) What is the dimension of NulA^T? Explain.

(3) (15 points) With as little computation as possible, determine which of the following is diagonalizable.

$$A = \begin{bmatrix} 3 & -7 & 9 & 5 \\ -7 & 1 & 1 & 2 \\ 9 & 1 & -2 & 1 \\ 5 & 2 & 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ 9 & 1 & -2 & 0 \\ 5 & 2 & 1 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -7 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -7 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix}.$$

(4) (15 points) Consider four polynomials

$$p_1 = 1 + t, \quad p_2 = -1 - t + t^2 + 2t^3, \quad p_3 = 1 - t + t^2 + 3t^3, \quad p_4 = t - t^2 - 3t^3$$

in P_3 and the subspace H of P_3 spanned by p_1, p_2, p_3.
1) Show that $\{p_1, p_2, p_3, p_4\}$ is a basis of P_3;
2) Which of $q = t^2 + t^3$ and $r = -1 + t + t^2 + t^3$ is in H. For the one in H, find the coordinates with respect to the basis $\{p_1, p_2, p_3\}$ of H;
3) Show that the restriction $D(p) = p' : H \to P_2$ of the derivative transformation on H is invertible.

(5) (17 points) Circle the right answers (no reasons needed, multiple answers possible)
1) Suppose a linear transformation $T : M(2,3) \to P_3$ is onto. Then the dimension of the kernel of T is

$$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6$$

2) Suppose u, v and $u+v$ are nonzero. Then $\text{Proj}_{u+v}(x+y)$ is equal to

$\text{Proj}_u(x+y) + \text{Proj}_v(x+y)$
$\text{Proj}_u x + \text{Proj}_v y$
$\text{Proj}_{u+v} x + \text{Proj}_{u+v} y$
$\text{Proj}_u x + \text{Proj}_v x + \text{Proj}_u y + \text{Proj}_v y$

3) Suppose A is a symmetric 3×3 matrix, with $\det(A - \lambda I) = (\lambda - 1)(\lambda + 2)^2$. Suppose $(1,2,3)$ and $(1,1,1)$ are eigenvectors with eigenvalue -2. Then the following are also eigenvectors of A

$$(-1,2,-1) \qquad (3,2,1) \qquad (-1,-2,-3) \qquad (1,2,1) \qquad (2,1,2) \qquad (0,1,2)$$

4) Suppose the solution of the equation $A_{5\times 7}x = 0$ contains two free variables. Then the following are true

$$\text{rank}A = 2 \qquad \text{dimNul}A = 2 \qquad \text{dimNul}A^T = 2$$

$$\text{rank}A = 3 \qquad \dim\text{Nul}A = 3 \qquad \dim\text{Nul}A^T = 3$$

$$\text{rank}A = 5 \qquad \dim\text{Nul}A = 5 \qquad \dim\text{Nul}A^T = 5$$

(6) (23 points) True or False (no reason needed)

1) If $Ax = 0$ has only trivial solution, then the columns of A is a basis of Col$A$
2) If $Ax = 0$ has only trivial solution, then the rank of A is the number of columns of A
3) If $Ax = 0$ has only trivial solution, then the rank of A is the number of rows of A
4) If $Ax = b$ has solution for all b, then the columns of A is a basis of Col$A$
5) If $Ax = b$ has solution for all b, then the rank of A is the number of columns of A
6) If $Ax = b$ has solution for all b, then the rank of A is the number of rows of A
7) If $\dim V = n$ and $\mathcal{B}$ spans V, then $\mathcal{B}$ contains at least n vectors
8) If $\mathcal{B}$ spans V and contains n vectors, then $\dim V \geq n$
9) If a linear transformation $T : V \to W$ is onto, then $\dim V \geq \dim W$
10) If a linear transformation $T : V \to W$ is one-to-one, then $\dim V \geq \dim W$
11) If a linear transformation $T : V \to W$ is invertible, then $\dim V = \dim W$
12) If A is invertible and diagonalizable, and B is not diagonalizable, then AB is not diagonalizable
13) Any square matrix has at least one eigenvector
14) There is a square matrix with no eigenvector
15) All symmetric matrices are diagonalizable
16) All diagonalizable matrices are symmetric
17) If $A_{n\times n}$ has n distinct eigenvalues, then A is diagonalizable
18) If $A_{n\times n}$ is diagonalizable, then A has n distinct eigenvalues
19) Linearly independent eigenvectors have distinct eigenvalues
20) Vectors in Col$A$ are orthogonal to vectors in Nul$A$
21) Vectors in Row$A$ are orthogonal to vectors in Nul$A$
22) Vectors in Col$A$ are orthogonal to vectors in Row$A$
23) For any A, A^TA is always diagonalizable

Answer to Math 111 Final, Autumn 1999, section 1

(1) The eigenvalues are 1 and -1. For $\lambda_1 = 1$, we find only one eigenvector $v_1 = (0, 0, 1)$. For $\lambda_2 = -1$, we find another eigenvector $v_2 = (0, 1, -1)$. Since we do not have three eigenvectors, A is not diagonalizable. In other words, we cannot write $A = PDP^{-1}$.

From $u = 2v_1 + 3v_2$, we have $A^n u = 2\lambda_1^n v_1 + 3\lambda_2^n v_2 = \begin{cases} 2v_1 + 3v_2 = (0, 3, -1) & \text{for even } n \\ 2v_1 - 3v_2 = (0, -3, 5) & \text{for odd } n \end{cases}$.

(2) The row operation reduces $[A, w]$ to

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & -5 & 2 & 1 \\ 0 & 0 & 0 & 0 & a-1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first two columns $x_1 = (1, -2, -1, 0, 3)$ and $x_2 = (0, 1, 1, 1, -1)$ are pivot columns of A, they form a bases of Col$A$. Then

$$u_1 = x_1 = (1, -2, -1, 0, 3), \quad u_2 = x_2 - \text{Proj}_{u_1} x_2 = \frac{1}{5}(2, 1, 3, 10, 1)$$

is an orthogonal basis of Col$A$. By the usual formula for the orthogonal projection, we find $\text{Proj}_{\text{Col}A} v = (1, 0, 1, 2, 1)$.

From the row operation above, w is in Col$A$ if and only if $a = 1$.

Since NulA^T is the orthogonal complement of Col$A$, we have $\text{Proj}_{\text{Nul}A^T} v = v - \text{Proj}_{\text{Col}A} v = (-1, 0, -1, 1, 0)$, and $\text{Proj}_{\text{Nul}A^T}(-2v) = -2\text{Proj}_{\text{Nul}A^T} v = (2, 0, 2, -2, 0)$.

The maximal number of linearly independent rows of A is the rank of A. From the row operation above, this number is 2.

The dimension of NulA^T is $5-2=3$, where 5 is the number of rows and 2 is the rank.

(3) A is diagonalizable because it is symmetric.

B has four distinct eigenvalues 3, 1, -2, -4. Therefore it is diagonalizable.

The characteristic equation of C is $[(\lambda-3)(\lambda-1)-(-7)(-7)][(\lambda+2)(\lambda+4)-(1)(1)] = (\lambda^2-4\lambda-46)(\lambda^2+6\lambda+7)=0$. It is then easy to see that the equation has four distinct solutions. Therefore C is diagonalizable.

The characteristic equation of D is $[(\lambda-3)(\lambda-1)-(-7)(-7)][(\lambda+2)(\lambda+4)-(-1)(1)] = (\lambda^2-4\lambda-46)(\lambda^2+6\lambda+9)=0$. We have three eigenvalues $2\pm\sqrt{50}$, 4. All of them has one eigenvector each. Since we do not have enough eigenvectors, D is not diagonalizable.

(4) By the usual translation $a+bt+ct^2+dt^3 \leftrightarrow (a,b,c,d)$, we carry out the row operation for $[p_1,p_2,p_3,p_4,q,r]$:

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & 2 & 3 & -3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -2 & 0 \end{bmatrix}$$

From the first four columns, we see that all columns in $[p_1,p_2,p_3,p_4]$ is pivotal. Therefore $\{p_1,p_2,p_3,p_4\}$ is a basis of P_3.

From the 1st, 2nd, 3rd, and the 5th columns, we find that q is not in the span H of $\{p_1,p_2,p_3\}$. From the 1st, 2nd, 3rd, and the 6th columns, we find that r is in the span H of $\{p_1,p_2,p_3\}$. Continuing this row operation, we further find the solution $r=2p_1+2p_2-p_3$. Therefore the coordinates of r with respect to the basis $\{p_1,p_2,p_3\}$ of H is $(2,2,-1)$.

The linear transformation D is between two vector spaces of the same dimension 3. Therefore we only need to show D to be onto in order for D to be an isomorphism. Since $p\in H$, we have $p=c_1p_1+c_2p_2+c_3p_3$. Then we need to show $D(p)=c_1p_1'+c_2p_2'+c_3p_3' = c_1(1)+c_2(-1+2t+4t^2)+c_3(-1+2t+6t^2) = b_1+b_2t+b_3t^2$ has solution for all b_1, b_2, b_3. By the usual translation $a+bt+ct^2 \leftrightarrow (a,b,c)$, we carry out the row operation for $[p_1',p_2',p_3']$:

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Since there is no row like $[0,0,0]$ in the row echelon form, we see that $\{p_1',p_2',p_3'\}$ indeed span P_2. This implies that the solution always exist.

Remark: The isomorphism can also be proved by showing H is one-to-one, or more precisely, $H(p)=0$ and $p\in H$ implies $p=0$.

(5) 1) The dimension of the kernel is $\dim M(2,3)-\dim P_3=2$

2) Only the third one is correct

3) The eigenvectors of A are either orthogonal to $(1,2,3)$ and $(1,1,1)$ (so eigenvalue$=1$), or a linear combination of $(1,2,3)$ and $(1,1,1)$ (so eigenvalue$=-2$). By these criteria, we find

$$(-1,2,-1) \qquad (3,2,1) \qquad (-1,-2,-3) \qquad (0,1,2)$$

are eigenvectors of A

4) The condition means dimNul$A=2$. Then rank$A=7-2=5$, and dimNul$A^T=5-5=0$

(6) T, T, F, F, F, T

T, F, T, F, T, F

T, F, T, F, T, F

F, F, T, F, T

Math 111 Final, Autumn 1999, section 2

(1) (20 points) Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 1 & 2 & -2 \\ 1 & -2 & -2 & 1 \\ 4 & 1 & 3 & 2 \end{bmatrix}, \qquad x = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

1) Determine whether x and y are in Col$A$
2) Find a basis of Nul$A$
3) Find the dimensions of Nul$A$, NulA^T, Row$A$, Col$A$
4) Find the orthogonal projection and the distance of $v = (1, 2, 3, 4)$ to Row$A$

(2) (15 points) Consider the quadratic form $Q(x_1, x_2, x_3) = 3x_2^2 + 4x_1x_2 - 2x_1x_3 - 4x_2x_3$
1) Find symmetric matrix A such that $Q(x) = x^T Ax$
2) Find orthogonal matrix U, such that the change of variable $x = Uy$ transforms Q into a quadratic form without cross-product term

(3) (15 points) Consider the matrix

$$A = \begin{bmatrix} -1 & a & b \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Find the condition on a and b so that A is diagonalizable. In the diagonalizable case, find invertible P and diagonal D, such that $A = PDP^{-1}$.

(4) (20 points) Consider vectors

$$p_1 = 1 + 2t + t^2, \qquad p_2 = 1 + 3t, \qquad p_3 = 1 + 2t^2$$

of the space P_2 of polynomials of degree ≤ 2. Consider a linear transformation $T : P_2 \to \mathbf{R}^3$ satisfying

$$T(p_1) = (1, 0, 1), \qquad T(p_2) = (0, 1, 1), \qquad T(p_3) = (1, -1, 0).$$

1) Show that p_1, p_2, p_3 form a basis of P_2
2) Find the coordinates of the monomials 1, t and t^2 with respect to the basis $\{p_1, p_2, p_3\}$
3) Find formula for $T(a + bt + ct^2)$
4) Is T onto? Is T one-to-one? Explain
5) Can you find an onto but not one-to-one linear transformation from P_2 to $\mathbf{R}^3$?
6) Can you find an onto and one-to-one linear transformation from P_2 to $\mathbf{R}^3$?

(5) (30 points) True or False (no reason needed)
1) If $u_1 \perp v_1$ and $u_2 \perp v_2$, then $u_1 + u_2 \perp v_1 + v_2$
2) If $u \perp v$, then for any scalars a and b, $au \perp bv$
3) If a subspace W is the orthogonal complement of another subspace V, then V is also the orthogonal complement of W
4) If A has an orthogonal basis of eigenvectors, then A^T also has an orthogonal basis of eigenvectors
5) If A and B have orthogonal basis of eigenvectors, then AB also has an orthogonal basis of eigenvectors
6) If A and B have orthogonal basis of eigenvectors, then $A + B$ also has an orthogonal basis of eigenvectors
7) Rows of U are orthogonal if and only if $U^T U$ is diagonal
8) Columns of U are orthogonal if and only if $U^T U$ is diagonal
9) If T is a linear transformation, and $T(v_1), T(v_2), \cdots, T(v_k)$ are linearly independent, then $v_1, v_2, \cdots, v_k$ are linearly independent
10) If $v_1, \cdots, v_k$ span a subspace H, then $v_1, \cdots, v_k$ is a basis of H
11) If $v_1, \cdots, v_k$ are linearly independent vectors in a subspace H, then $v_1, \cdots, v_k$ is a basis of H
12) If all eigenvalues of A are non-zero, then A is invertible
13) If A is invertible, then all eigenvalues of A are non-zero
14) If rank$A_{m\times n} = n$, then columns of A span $\mathbf{R}^m$

15) If $\operatorname{rank} A_{m\times n} = n$, then columns of A are linearly independent

Answer to Math 111 Final, Autumn 1999, section 2

(1) Row operation on $[A, x, y]$ gives

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore x is in $\mathrm{Col}A$ and y is not in $\mathrm{Col}A$.

By solving $Ax = 0$ (using first four columns of the row operation, we find a basis $u = (-1, -1, 1, 1)$ of $\mathrm{Nul}A$. The row operation also tells us the rank of A is 3. Therefore, $\dim \mathrm{Nul}A = 1$, $\dim \mathrm{Nul}A^T = 5-3 = 2$, $\dim \mathrm{Row}A = \dim \mathrm{Col}A = 3$.

Using the fact that $\mathrm{Row}A$ is an orthogonal complement of $\mathrm{Nul}A$, we have

$$\begin{aligned} \mathrm{Proj}_{\mathrm{Nul}A} v &= \frac{v \cdot u}{u \cdot u} u = \frac{4}{4}(-1, -1, 1, 1) = (-1, -1, 1, 1) \\ \mathrm{dist}(v, \mathrm{Row}A) &= ||\mathrm{Proj}_{\mathrm{Nul}A} v|| = 2 \\ \mathrm{Proj}_{\mathrm{Row}A} v &= v - \mathrm{Proj}_{\mathrm{Nul}A} v = (2, 3, 2, 3) \end{aligned}$$

(2) The symmetric matrix A is

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$$

It has two eigenvalues -1, 5. For $\lambda_1 = -1$, we solve $(A + I)v = 0$ to find eigenvectors $v_1 = (1, 0, 1)$, $v_2 = (-2, 1, 0)$. For $\lambda_1 = 5$, we solve $(A - 5I)v = 0$ to find eigenvector $v_3 = (-1, -2, 1)$. We use the Gram-Schmidt process to orthogonalize v_1, v_2 and get

$$u_1 = v_1 = (1, 0, 1), \quad u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = (-1, 1, 1).$$

Then we normalize the orthogonal basis u_1, u_2, v_3 to get the orthogonal matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The change of variable $x = Uy$ transforms Q into $Q(Uy) = -y_1^2 - y_2^2 + 5y_3^2$.

(3) The characteristic equation is $(-1-\lambda)[(1-\lambda)^2 - 2^2] = -(\lambda - 3)(\lambda + 1)^2$. For eigenvalue $\lambda_1 = 3$, we find exactly one eigenvector $(\frac{a+b}{4}, 1, 1)$. For eigenvalue $\lambda_1 = -1$, we need to solve the system $ax_2 + bx_3 = 0, 2x_2 + 2x_3 = 0$, which already has x_1 as one free variable. In order to diagonalize, we have to have another free variable among x_2 and x_3. The condition for this is $a = b$. In this case, the general solution of the homogeneous equation is $x_1(1, 0, 0) + x_3(0, -1, 1)$, and we get two eigenvectors $(1, 0, 0)$ and $(0, -1, 1)$.

Thus A is diagonalizable if and only if $a = b$. In this case, $A = PDP^{-1}$ for (note that $\frac{a+b}{4} = \frac{a}{2}$)

$$P = \begin{bmatrix} \frac{a}{2} & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(4) Using the translation $a + bt + ct^2 \leftrightarrow (a, b, c)$, we carry out row operation on $[p_1, p_2, p_3, 1, t, t^2]$ as follows

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row op.}} \begin{bmatrix} 1 & 0 & 0 & -6 & 2 & 3 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{bmatrix},$$

From the first three columns, we see that p_1, p_2, p_3 form a basis of P_2 The row operation also gives us the coordinates of the monomials 1, t and t^2 with respect to the basis $\{p_1, p_2, p_3\}$:

$$[1] = (-6, 4, 3), \qquad [t] = (2, -1, -1), \qquad [t^2] = (3, -2, -1).$$

In particular, we get

$$\begin{aligned} T(1) &= -6T(p_1) + 4T(p_2) + 3T(p_3) = (-3, 1, 2) \\ T(t) &= 2T(p_1) - T(p_2) - T(p_3) = (1, 0, 1) \\ T(t^2) &= 3T(p_1) - 2T(p_2) - T(p_3) = (2, -1, 1) \end{aligned}$$

Therefore

$$T(a + bt + ct^2) = a(-3, 1, 2) + b(1, 0, 1) + c(2, -1, 1) = (-3a + b + 2c, a - c, 2a + b + c).$$

We may translate and get the matrix of T. The row operation

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row op.}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

implies that T is not onto (there is $[0, 0, 0]$) and not one-to-one (the third column is not pivot).

Since both P_2 and $\mathbf{R}^3$ have the same dimension 3, a linear transformation between them is onto if and only if it is one-to-one. In particular, we cannot find an onto but not one-to-one linear transformation from P_2 to $\mathbf{R}^3$. The same dimension also implies we can find an onto and one-to-one linear transformation from P_2 to $\mathbf{R}^3$. $T(a + bt + ct^2) = (a, b, c)$ is such an example.

(5)

F, T, T, T, F
T, F, T, T, F
F, T, T, F, T