OLD MATH111 MIDTERM EXAMS AND ANSWERS

Note1: The answers are not guaranteed to be correct. Please report any mistakes to me

Note2: The old midterm exams may cover different materials. Not all the past problems are suitable for the new midterm exam

Spring 1995

(1) (15 points) Consider

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & -2 & 0 & -1 \end{bmatrix}.$$

1) Find $\det A$;
2) Find $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$;
3) Find $\det(AA^T)$, $\det(A^3)$;
4) Are first three columns of A linearly independent?

(2) (10 points) Consider the linear transformation

$$T(x_1, x_2, x_3) = (-2x_2 + x_3, x_1 - x_3, 3x_1 + x_2 - 3x_3).$$

1) Find the inverse linear transformation T^{-1};
2) Find x such that $T(x) = (-1, -2, -4)$.

(3) (15 points) Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^4$ given by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$$

1) Is T onto?
2) Is T one-to-one?
3) For what h is the vector $(0, -1, h, 1)$ in the image of T?

(4) (15 points) Consider

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

1) Show that AB is an upper triangular matrix;
2) Are A, B invertible? (*Hint: Use the computation in part 1*)
3) Are columns of A linearly independent?
4) Do columns of B span $\mathbf{R}^4$.

(5) (15 points) Are the following subspaces? You need to provide reason for your conclusion.

1) $H = \{(x, y, z) : x \geq y \geq z\} \subset \mathbf{R}^3$;
2) $H = \{(2a + b, b, a - 3b) : a,b \text{ are integers}\} \subset \mathbf{R}^3$;
3) $H = \{(u, v, w) : u - v = v - w = w - u\} \subset \mathbf{R}^3$;
4) $H = \{p(t) : 3p(t) = tp'(t)\} \subset \mathbf{P}_5$.
($\mathbf{P}_5$ is the space of polynomials of degree ≤ 5, and $p'(t)$ is the derivative of $p(t)$)

(6) (15 points) Consider

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 1 & -3 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 & 2 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}.$$

1) Show that B is the inverse of A;
2) Compute

$$C = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} B;$$

3) Compute C^{1995}.

(7) (15 points)
True or False
(no reason needed, in statements 7, 8, and 9, the matrices are of size $n \times n$)
1) If v_1, v_2, v_3, v_4 are linearly independent, then v_1, v_2, v_3 are linearly independent;
2) If v_1, v_2, v_3, v_4 are linearly dependent, then v_1, v_2, v_3 are linearly dependent;
3) If v_1, v_2, v_3, v_4 span V, then v_1, v_2, v_3 span V;
4) If v_1, v_2, v_3, v_4 do not span V, then v_1, v_2, v_3 do not span V;
5) $v_1, v_2, v_3, v_4 \in \mathbf{R}^3$ always span $\mathbf{R}^3$;
6) $v_1, v_2, v_3 \in \mathbf{R}^4$ never span $\mathbf{R}^4$;
7) If ABC is invertible, then A, B, C are invertible;
8) $(AB)^2 = A^2B^2$;
9) If A, B are invertible, then $(AB)^{-1} = A^{-1}B^{-1}$;
10) If A is a 5×5 matrix, then $\det(2A)=2\det A$;
11) If A is a 5×5 matrix, then $\det(-A) = -\det A$;
12) If A is a 5×5 matrix and $Ax = 0$ has no nonzero solution, then the columns of A span $\mathbf{R}^5$;
13) If A is a 5×5 matrix and the columns of A are linearly dependent, then $Ax = b$ has infinitely many solution for any $b \in \mathbf{R}^5$.
14) If $Ax = b$ has more than one solution, then $Ax = 0$ has more than one solution;
15) if $Ax = 0$ has more than one solution, then $Ax = b$ has more than one solution.

Sorry, I lost my answer file for Midterm of Spring 1995
Autumn 1995

(1) (10 points) Consider

$$A = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

1) Find a special solution of $Ax = b$ satisfying $x_1 = x_2 = 0$;
2) Solve the homogenuous equation $Ax = 0$;
3) Solve the equation $Ax = b$.

(2) (15 points) Consider

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Find the indicated matrix if it is meaningful:

$$AB, \quad BA, \quad A + B, \quad AC, \quad BC, \quad (AB)^{-1}, \quad C^{-1}, \quad .$$

(3) (10 points) Consider

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 3 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ h \\ h \end{bmatrix}.$$

1) Does $Ax = 0$ have nontrivial solution?
2) For what h, does $Ax = b$ have solution?

3) What is det$A$?
4) Are columns of A linearly independent?
5) Do columns of A span $\mathbf{R}^4$?

Please provide reason.

(4) (10 points) Consider the linear transformation

$$T(x_1, x_2) = (2x_1 + x_2, 3x_1 + 2x_2, x_1 - x_2).$$

1) Is T onto?
2) Is T one-to-one?

Please provide reason.

(5) (10 points) Consider the linear transformation

$$T(x_1, x_2, x_3) = (x_3, x_2 + 2x_3, x_1 + 2x_2 + 3x_3).$$

1) Is T invertible? If no, provide reason. If yes, find T^{-1}.
2) Find x such that $T(x) = (3, 2, 1)$.

(6) (15 points) Which of the following are linearly independent? Which of the following span $\mathbf{R}^4$? Please provide reason.

1)

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix};$$

2)

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix};$$

3)

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

(7) (15 points) Consider

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

1) Compute det$A$, detA^{100}, $\det(A^T A^{-3} A^T)$;
2) Find the inverse of A;
3) Find the inverse of A^2;
4) Use A^{-1} to solve equation $Ax = b$.

(8) (15 points) True or False (*no reason needed, in statements 7, 8, and 9, the matrices are of size $n \times n$*)

1) If $v_1, v_2, \cdots, v_p$ span V, then $v_1, v_2, \cdots, v_p, v_{p+1}$ span V;
2) If $v_1, v_2, \cdots, v_p$ do not span V, then $v_1, v_2, \cdots, v_p, v_{p+1}$ do not span V;
3) If any two of v_1, v_2, v_3 are linearly independent, then the three vectors v_1, v_2, v_3 are linearly independent;
4) If the three vectors v_1, v_2, v_3 are linearly independent, then any two of v_1, v_2, v_3 are linearly independent;
5) If the vectors $v_1, v_2, v_3, v_4 \in \mathbf{R}^4$ are linearly dependent, then the 4×4 matrix $A = [v_1, v_2, v_3, v_4]$ is not invertible;
6) If the 4×4 matrix $A = [v_1, v_2, v_3, v_4]$ is not invertible, then the vectors $v_1, v_2, v_3, v_4 \in \mathbf{R}^4$ are linearly dependent;

7) If AB is invertible, then BA is invertible;

8) $(A+B)(A-B) = A^2 - B^2$;

9) If $A^2 = 0$, then $A = 0$;

10) If the first row is moved to second row, the second to third, and the third to first, then such "triple row exchanges" change determinant by multiplying -1;

11) $\det(2A) = 2\det A$;

12) If A is a 4×4 matrix and $Ax = 0$ has nontrival solution, then the columns of A do not span $\mathbf{R}^4$;

13) If A is a 4×4 matrix and the columns of A are linearly independent, then $Ax = b$ has solution for any $b \in \mathbf{R}^4$.

14) If A is a 4×5 matrix, then $Ax = b$ always has solution;

15) If A is a 4×5 matrix, then $Ax = 0$ has nontrivial solution.

Answer to Midterm of Autumn 1995

(1) $x_s = (0,0,1,0)$ is the obvious solution. The solution to homogenuous equation is $x_g = x_3(-1,2,1,0)$. The general solution to $Ax = b$ is then $(0,0,1,0) + c(-1,2,1,0) = (-c, 2c, 1+c, 0)$.

(2) $AB = \begin{bmatrix} -1 & 3 \\ 1 & -5 \end{bmatrix}$. $BA = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -5 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. $BC = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$.

$(AB)^{-1} = \frac{1}{2}\begin{bmatrix} -5 & -3 \\ -1 & -1 \end{bmatrix}$. The others are meaningless.

(3) $Ax = 0$ has nontrivial solution. The condition for $Ax = b$ to have solution is $h = \frac{3}{2}$. $\det A = 0$. The columns of A are linearly dependent, and they do not span $\mathbf{R}^4$.

(4) T is not onto because it maps 2-dim to 3-dim. T is one-to-one because $(2,3,1)$ and $(1,2,-1)$ are clearly linearly independent.

(5) $T^{-1}(x_1,x_2,x_3) = (x_1 - 2x_2 + x_1, -2x_1 + x_2, x_1)$. $x = T^{-1}(3,2,1) = (3-2\cdot 2+1, -2\cdot 3+2, 3) = (0,-4,3)$.

(6) 1) independent (not parallel), do not span (too few vectors); 2) dependent, do not span (determinant=0); 3) dependent (too many vectors), span (first four already span).

(7) $\det A = -1$, $\det A^{100} = (-1)^{100} = 1$, $\det(A^T A^{-3} A^T) = \det A^{-1} = -1$.

$$A^{-1} = \begin{bmatrix} -3 & 2 & -3 & -6 \\ 2 & -1 & 2 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & -1 & -2 \end{bmatrix}, \ (A^2)^{-1} = (A^{-1})^2 = \begin{bmatrix} 13 & -8 & 28 & 53 \\ -8 & 5 & -18 & -34 \\ 0 & 0 & 14 & 25 \\ 0 & 0 & 5 & 9 \end{bmatrix}.$$

solution$= A^{-1}b = (-2,1,2,1)$.

(8) 1) T; 2) F; 3) F; 4) T; 5) T; 6) T; 7) F; 8) F; 9) F; 10) F; 11) F; 12) T; 13) T; 14) F; 15) T.

Spring 1997

(1) (17 points) Consider

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ -2 & 2 & -1 & 4 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}.$$

1) For what h, does $Ax = b$ have solution?

2) Find general solution of homogeneous equation $Ax = 0$.

3) Do columns of A span $\mathbf{R}^3$? Explain.

4) Can you find three columns of A that are linearly independent? Explain.

(2) (17 points) Consider the linear transformation

$$T(x_1,x_2,x_3) = (x_1 + 2x_3, 2x_1 - x_2 + 3x_3, 4x_1 - x_2 + 8x_3).$$

1) Find the matrix A corresponding to T;

2) Find the determinant of A, A^T and $2A$;

3) Find the inverse of A and the determinant of A^{-1};

4) Find x such that $T(x) = (-1, -2, 0)$.

(3) (16 points) Are the following subspaces? You need to provide reason for your conclusion.

1) $H = \{(x, y, z) : x - y = 2z, x + 2y = 0\} \subset \mathbf{R}^3$;

2) $H = \{(x + y, x - y, xy) : x,\ y \text{ in } \mathbf{R}\} \subset \mathbf{R}^3$;

3) $H = \{(x, y, z) : x + y = 2y - z + 1 = 3x - z\} \subset \mathbf{R}^3$;

4) $H = \{p(t) : p(0) = p(1) = p''(2)\} \subset \mathbf{P}_5$.

($\mathbf{P}_5$ is the space of polynomials of degree ≤ 5, and $p''(t)$ is the second order derivation of $p(t)$)

(4) (16 points) Which of the following are linearly independent? Which of the following span $\mathbf{R}^3$? Please provide reason.

1)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix};$$

2)

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix};$$

3)

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \sqrt{5} \\ 101 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}.$$

4)

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 11 \\ 111 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0.3 \\ 0.03 \end{bmatrix}.$$

(5) ($2 \times 17 = 34$ points) True or False (no reason needed)

in $1 - 5$, v_1, v_2, v_3, v_4, v_5 *are vectors in* $\mathbf{R}^3$

1) v_1, v_2, v_3, v_4, v_5 always span $\mathbf{R}^3$;

2) v_1, v_2, v_3, v_4, v_5 are always linearly dependent;

3) If v_1, v_2, v_3 span $\mathbf{R}^3$, then v_1, v_2, v_3 are linearly independent;

4) If v_1, v_2, v_3 do not span $\mathbf{R}^3$, then v_1, v_2, v_3 are not linearly independent;

5) If v_1 and v_2 are not multiples of v_3, then v_1, v_2, v_3 are linearly independent;

in $6 - 11$, A *is a* 4×6 *matrix,* B *and* C *are* 6×6 *matrices*

6) The null space of A is a subspace of $\mathbf{R}^4$;

7) The column space of A is a subspace of $\mathbf{R}^4$;

8) $Ax = 0$ can never have unique solution;

9) If $Bx = 0$ has unique solution, then $Bx = b$ always have solution;

10) If columns of B span $\mathbf{R}^6$, then $Bx = b$ always have solution;

11) If $BC = I$, then $CB = I$;

12) $\det(B + C) = \det B + \det C$;

13) $\det(-B) = -\det B$;

in $14 - 17$, $T : \mathbf{R}^3 \to \mathbf{R}^5$ *and* $S : \mathbf{R}^5 \to \mathbf{R}^5$ *are linear transformations*

14) If T and S are one-to-one, then ST is one-to-one;

15) If S is one-to-one, then S is onto;

16) T is never one-to-one;

17) T is never onto.

Answer to Midterm of Spring 1997

(1) After row operation, $[A, b]$ becomes

$$\begin{bmatrix} 1 & -1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & h+1 \end{bmatrix}$$

1) $h = -1$.
2) $x_1 = x_2 + 3x_4$, $x_3 = -2x_4$, x_2 and x_4 are free variables.
3) No. By 1), $Ax = b$ does not always have solution.
4) No. Three columns linearly independent would imply the three columns span $\mathbf{R}^3$. Thus columns of A span $\mathbf{R}^3$, contradicting 3).

(2)
1)

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & -1 & 8 \end{bmatrix};$$

2) $\det A = -1$, $\det A^T = \det A = -1$ and $\det(2A) = 2^3 \det A = -8$;
3)

$$A^{-1} = \begin{bmatrix} 5 & 2 & -2 \\ 4 & 0 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

$\det A^{-1} = (\det A)^{-1} = -1$;
4)

$$A^{-1} \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ -4 \\ 4 \end{bmatrix}.$$

Hence $x = (-9, -4, 4)$.

(3)
1) Yes. Solution to homogeneous equation $x - y - 2z = 0, x + 2y = 0$;
2) No. If $2(x + y, x - y, xy) = (2x + 2y, 2x - 2y, 2xy)$ were in H, then $2xy = (2x)(2y) = 4xy$, contradiction;
3) No. Not containing $(0, 0, 0)$;
4) Yes. $p(0) = p(1)$ and $q(0) = q(1)$ implies $(p + q)(0) = p(0) + q(0) = p(1) + q(1) = (p + q)(1)$, $(cp)(0) = c(p(0)) = c(p(1)) = (cp)(1)$. Similar for the equation $p(1) = p''(2)$.

(4)
1) Independent because not parallel. Not span $\mathbf{R}^3$ because $2 < 3$.
2) Independent and span $\mathbf{R}^3$ because determinant$= -2 \neq 0$.
3) Dependent because first and third are parallel. Not span $\mathbf{R}^3$ because $3 = 3$ and dependent.
4) Dependent because $4 > 3$. Span $\mathbf{R}^3$ because the first three vectors span $\mathbf{R}^3$.

(5)
1) F; 2) T; 3) T; 4) T; 5) F; 6) F; 7) T; 8) T; 9) T; 10) T; 11) T; 12) F; 13) F; 14) T; 15) T; 16) F; 17) T.

Autumn 1997

(1) (18 points) Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & a \\ 0 & 1 & a & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

1) Find conditions on a, so that $Ax = b$ has solution.
2) Can you find a, such that $Ax = b$ has a unique solution? Explain.
3) For $a = 2$, find the general solution of $Ax = b$.

4) Find conditions on a, so that the columns of A span $\mathbf{R}^3$.

5) Can you find a, such that the columns of A are linearly independent? Explain.

(2) (18 points) Consider the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^4$ given by the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & 3 \end{bmatrix}.$$

1) Find $\det A$, $\det(2A)$, $\det A^2$;

2) Explain why A is invertible and then find the inverse A^{-1};

3) Find $\det A^{-1}$ and $\det(AA^T A^{-3})$;

4) Write down the formula for T and T^{-1};

5) What is the range of T?

(3) (15 points) Are the following subspaces? You *do not* need to provide reason if S *is* a subspace. You *do* need to provide reason if S *is not* a subspace.

1) $S = \{(x, y, z) : x^2 - y^2 + z^2 = 0, x - y + z = 0\} \subset \mathbf{R}^3$;

2) $S = \{(x + y, y, x + z) : x + 2y - z = 0\} \subset \mathbf{R}^3$;

3) $S = \{A : A^t + 2A = O\} \subset M(4, 4)$.

(O *means the zero matrix*)

(4) (15 points) Are the following linear transformations? You *do not* need to provide reason if L *is* a linear transformation. You *do* need to provide reason if L *is not* a linear transformation.

1) $L(x, y, z) = (x + y, x - z) : \mathbf{R}^3 \to \mathbf{R}^2$;

2) $L(x, y, z) = (x + y - 1, x - z + 1) : \mathbf{R}^3 \to \mathbf{R}^2$;

3) $L(p) = \begin{bmatrix} p(0) & p'(0) \\ p(1) & p'(1) \end{bmatrix} : P^3 \to M(2, 2)$.

(5) (18 points) Consider the polynomials

$$p_1 = 1 + 2x + 3x^3, \quad p_2 = 2 + x + 3x^2, \quad p_3 = 1 + x + x^2 + x^3$$

in the space P^3 of polynomials of degree ≤ 3.

1) Find the conditions for a polynomial $p(x) = a + bx + cx^2 + dx^3$ to be in $\text{span}(p_1, p_2, p_3)$;

2) Are p_1, p_2, p_3 linearly independent? If not, write one polynomial as a linear combination of the other two;

3) Find two other polynomials q_1, q_2, such that p_1, p_2, p_3, q_1, q_2 span P^3.

(6) (16 points) True or False (no reason needed)

in $1 - 5$, v_1, v_2, v_3, v_4 *are vectors in* $\mathbf{R}^5$

1) v_1, v_2, v_3, v_4 can never span $\mathbf{R}^5$;

2) v_1, v_2, v_3, v_4 can never be linearly independent;

3) If v_1, v_2, v_3, v_4 are linearly dependent, then at least two vectors from v_1, v_2, v_3, v_4 are parallel;

4) If v_1, v_2, v_3, v_4 are linearly independent, then v_1, v_2, v_3 are linearly independent;

5) If v_1, v_2, v_3, v_4 are linearly dependent, then v_1, v_2, v_3 are linearly dependent;

in $6 - 15$, A *can be any* 6×5 *matrix,* B *and* C *can be any* 5×5 *matrices*

6) The null space of A is a subspace of $\mathbf{R}^5$;

7) The range space of A is a subspace of $\mathbf{R}^5$;

8) If columns of A are linearly independent, then the rows of A span $\mathbf{R}^5$;

9) If columns of B span $\mathbf{R}^5$, then the rows of B span $\mathbf{R}^5$;

10) One can always find a vector b, such that the equation $Ax = b$ does not have solution;

11) $Ax = 0$ has nontrivial (i.e., $x \neq 0$) solution;

12) If BC is invertible, then B and C are invertible;

13) $A(B - C) = AB - AC$;

14) If $AB = AC$, then either $A = 0$ or $B = C$;

15) $\det(-B) = -\det B$;

16) $B^2 = C^2 \Longrightarrow B = C$ or $B = -C$;

Answer to Midterm of Autumn 1997

(1) After row operation, $[A, b]$ becomes

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1-a & 0 \\ 0 & 0 & a-1 & a-1 & 1 \end{bmatrix}$$

1) The condition for the solution to exist is $a \neq 1$.

2) $Ax = b$ can never have unique solution, because the number of varibles is more than the number of equations (so some column must be nonpivotal).

3) The general solution in case $a = 2$ is $x = (1, -1, 1, 0) + c(-2, 2, -1, 1)$.

4) The condition for the columns to span $\mathbf{R}^3$ is $a \neq 1$.

5) Columns of A can never be linearly independent, because there are more vectors than the dimension.

(2)

1) Expanding along first row, we have

$$\det A = (-1)^{1+3} 1 \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix}.$$

Expanding again along second row, we have

$$\det A = (-1)^{2+2}(-1) \det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = -1.$$

$\det(2A) = 2^4 \det A = -16$, $\det A^2 = (\det A)^2 = 1$.

2) A is invertible because $\det A \neq 0$. By performing row operation $[A, I] \to [I, A^{-1}]$, we obtain

$$A^{-1} = \begin{bmatrix} 8 & 3 & -1 & -1 \\ 3 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -8 & -2 & 1 & 1 \end{bmatrix}$$

3) $\det A^{-1} = (\det A)^{-1} = -1$. $\det(AA^T A^{-3}) = (\det A)^2 (\det A)^{-3} = -1$.

4)$T(x_1, x_2, x_3, x_4) = Ax = (x_3, x_1 + x_4, -x_2 + 3x_3, 2x_1 + x_2 + 5x_3 + 3x_4)$, $T^{-1}(x_1, x_2, x_3, x_4) = A^{-1}x = (8x_1 + 3x_2 - x_3 - x_4, 3x_1 - x_3, x_1, -8x_1 - 2x_2 + x_3 + x_4)$.

5) Since A is invertible, T is also invertible. In particular, T is onto. Therefore the range of T is $\mathbf{R}^4$.

(3)

1) Not subspace. $(1, 1, 0)$ and $(0, 1, 1)$ are in S. But $(1, 1, 0) + (0, 1, 1) = (1, 2, 1)$ does not satisfy $x^2 - y^2 + z^2 = 0$.

2) Yes. $(0, 0, 0) = (0+0, 0, 0+0)$ for $x = y = z = 0$, which satisfies $x + 2y - z = 0$. Moreover, suppose $u = (x + y, y, x + z)$ and $v = (x' + y', y', x' + z')$, where $x + 2y - z = 0$, and $x' + 2y' - z' = 0$. Then

$$u + v = ((x + x') + (y + y'), (y + y'), (x + x') + (z + z')), \quad cu = (cx + cy, cy, cx + cz).$$

Since

$$(x + x') + 2(y + y') - (z + z') = x + 2y - z + x' + 2y' - z' = 0,$$
$$cx + 2cy - cz = c(x + 2y - z) = 0,$$

we see that $u + v$ and cu are in S.

3) Yes. $O^t + 2O = O \Rightarrow O$ is in S. Moreover A and B are in $S \Rightarrow A^t + 2A = O$, $B^t + 2B = O \Rightarrow$

$$(A + B)^t + 2(A + B) = (A^t + 2A) + (B^t + 2B) = O + O = O, \quad (cA)^t + 2(cA) = c(A^t + 2A) = 2O = O,$$

$\Rightarrow A + B$ and cA are in S.

(4)

1) L is linear: $L(x+x', y+y', z+z') = ((x+x')+(y+y'), (x+x')-(z+z')) = (x+y, x-z)+(x'+y', x'-z') = L(x,y,z)+L(x',y',z')$, $L(cx,cy,cz) = (cx+cy, cx-cz) = c(x+y, x-z) = cL(x,y,z)$.

2) L is not linear: $L(0,0,0) = (-1,1)$, $L(2(0,0,0)) = (-1,1)$ is different from $2L(0,0,0) = 2(-1,1) = (-2,2)$.

3) L is linear (note $(p+q)'(x) = (p'+q')(x) = p'(x)+q'(x)$, $(cp)'(x) = cp'(x)$):

$$L(p+q) = \begin{bmatrix} (p+q)(0) & (p+q)'(0) \\ (p+q)(1) & (p+q)'(1) \end{bmatrix} = \begin{bmatrix} p(0)+q(0) & p'(0)+q'(0) \\ p(1)+q(1) & p'(1)+q'(1) \end{bmatrix} = \begin{bmatrix} p(0) & p'(0) \\ p(1) & p'(1) \end{bmatrix} + \begin{bmatrix} q(0) & q'(0) \\ q(1) & q'(1) \end{bmatrix}$$

$$= L(p)+L(q)$$

The verification of $L(cp) = cL(p)$ is similar.

(5) Use row operation,

$$\begin{bmatrix} 1 & 2 & 1 & a \\ 2 & 1 & 1 & b \\ 0 & 3 & 1 & c \\ 3 & 0 & 1 & d \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & a \\ 0 & 3 & 1 & c \\ 0 & 0 & 0 & -2a+b+c \\ 0 & 0 & 0 & -3a+2c+d \end{bmatrix}$$

1) The condition for p to be in the span is $-2a+b+c = -3a+2c+d = 0$.

2) p_1, p_2, p_3 are linearly dependent. From the row operations, we see $c_1p_1+c_2p_2+c_3p_3 = 0$ is the same as $c_1+2c_2+c_3 = 0$, $3c_1+c_3 = 0$. One solution is $c_1 = 1, c_2 = 1, c_3 = -3$. Thus we get $p_1+p_2-3p_3 = 0$, or $p_1 = 3p_3 - p_2$.

3) Try $q_1 = 1, q_2 = x$, we use row operation (the middle matrix is obtained by the replacement $(a,b,c,d) = (1,0,0,0)$ for q_1, and $(a,b,c,d) = (0,1,0,0)$ for q_2,

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Because there is no $(0,0,0,0,0)$, the polynomials span P^3.

(6)

1) T; 2) F; 3) F; 4) T; 5) F; 6) T; 7) F; 8) T; 9) T; 10) T; 11) F; 12) T; 13) T; 14) F; 15) T; 16) F;

Spring 1998

(1) Consider

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & a \\ 1 & 1 & 0 \\ 0 & a & -1 \end{bmatrix}, \quad b = \begin{bmatrix} a \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

1) For what a does the equation $Ax = b$ have no solution? unique solution? nonunique solution?

2) Find the general solution of $Ax = b$ in cases the equation has solutions.

3) Can columns of A span $\mathbf{R}^4$? Explain.

4) For what a, the columns of A are linearly independent?

(2) Consider the linear transformation

$$T(x_1,x_2,x_3,x_4) = (x_2+2x_3+4x_4, x_1+x_2+2x_3, x_1+x_2+2x_3-2x_4, x_1+x_2+x_3+2x_4).$$

1) Find the matrix A of $T : \mathbf{R}^4 \to \mathbf{R}^4$;

2) Find $\det A$;

3) Find formula for T^{-1};

4) Find $\det A^{-1}$, $\det A^TA^2(A^{-1})^T$, $\det(-A)$;

5) Find a vector v such that $T(v) = (1,0,0,0)$.

(3) Consider vectors $v_1 = (1,2,3,4)$, $v_2 = (2,3,4,1)$, $v_3 = (3,4,1,2)$.

1) Is $b = (0, 0, 1, -1)$ in the span of v_1, v_2, v_3?
2) Are v_1, v_2, v_3 linearly independent? Explain.
3) Do v_1, v_2, v_3 span $\mathbf{R}^4$? If not, how many more vectors do you need to add in order to span $\mathbf{R}^4$? Explain.

(4) Which of the following are subspaces? Explain.
1) $H = \{(a, b, c, d) : a + 2b = c = d + b - 1\} \subset \mathbf{R}^4$;
2) $H = \{(a + 2b, 3a - 5b, a - 2b) : \text{all } a, b\} \subset \mathbf{R}^3$;
3) $H = \{p(t) \in P_4 : p(0) = p(1)\} \subset P_4$;
4) $H = \{\text{non-invertible } 2 \times 2 \text{ matrices}\} \subset M(2, 2)$.

(5) True or False (*no reason needed*)
1) v_1, v_2, v_3 are independent, and v_3, v_4, v_5 are independent $\Longrightarrow v_1, v_2, v_3, v_4, v_5$ are independent;
2) v_1, v_2, v_3, v_4, v_5 are independent $\Longrightarrow v_1, v_2, v_3$ are independent, and v_3, v_4, v_5 are independent;
3) b is in the span of $v_1, v_2, v_3, v_4, v_5 \Longrightarrow b$ is in the span of v_1, v_2, v_3, v_4;
4) b is in the span of $v_1, v_2, v_3, v_4 \Longrightarrow b$ is in the span of v_1, v_2, v_3, v_4, v_5;
5) Columns of $A_{5\times6}$ span $\mathbf{R}^5 \Longrightarrow Ax = b$ has solution for all b;
6) Columns of $A_{5\times6}$ span $\mathbf{R}^5 \Longrightarrow Ax = 0$ has only trivial solution;
7) $A_{5\times6}x = 0$ always has nontrivial solution;
8) $A_{6\times5}x = 0$ always has nontrivial solution;
9) Columns of $A_{5\times6}$ always span $\mathbf{R}^5$;
10) Columns of $A_{6\times5}$ always span $\mathbf{R}^6$;
11) For square matrices A, B, C, $ABC = 0 \Longrightarrow$ one of A, B, C is 0;
12) A is invertible $\Longrightarrow A^2 \neq 0$;
13) $A^2 \neq 0 \Longrightarrow A$ is invertible;
14) Row operation changes A to $B \Longrightarrow \det A = \det B$;
15) Row operation changes A to $0 \Longrightarrow \det A = 0$.

Answer to Midterm of Spring 1998

(1) Row operations reduce $[A \quad b]$ to

$$\begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & 1 & a \\ 0 & 0 & -a-1 & -a-1 \\ 0 & 0 & 0 & -a^2+a+2 \end{bmatrix}$$

$Ax = b$ has no solution when $a \neq 2$ or -1; has unique solution when $a = 2$; has nonunique solution when $a = -1$. The general solution in case $a = -1$ is $x = (1, -1, 0) + x_3(1, -1, 1)$. The general solution in case $a = 2$ is $x = (-1, 1, 1)$. Since $3 < 4$, columns of A cannot span $\mathbf{R}^4$. From the result of row operation, columns of A are linearly independent when $a \neq -1$.

(2)

$$A = \begin{bmatrix} 0 & 1 & 2 & 4 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -1 & 3 & -2 & 0 \\ 1 & -6 & 4 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix}.$$

$T^{-1}(x_1, x_2, x_3, x_4) = (-x_1 + 3x_2 - 2x_3, x_1 - 6x_2 + 4x_3 + 2x_4, 2x_2 - x_3 - x_4, (1/2)x_2 - (1/2)x_3)$. $\det A = 2$, $\det A^{-1} = -1/2$, $\det A^T A^2 (A^{-1})^T = (\det A)^2 = 4$, $\det(-A) = \det A = 2$. $v = T^{-1}(1, 0, 0, 0) = (-1, 1, 0, 0)$.

Consider vectors $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$.
1) Is $b = (0, 0, 1, -1)$ in the span of v_1, v_2, v_3?
2) Are v_1, v_2, v_3 linearly independent? Explain.
3) Do v_1, v_2, v_3 span $\mathbf{R}^4$? If not, how many more vectors do you need to add in order to $\mathbf{R}^4$? Explain.

(3) Row operations reduce $[v_1 \quad v_2 \quad v_3 \quad b]$ to

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the first three (and only the first three) columns are pivot, b is in the span and v_1, v_2, v_3 are linearly independent. The three vectors cannot span $\mathbf{R}^4$. Since the three vectors are linearly independent, we need $4-3$ =one more vector in order to span $\mathbf{R}^4$.

(4)

1) not a subspace because not contain zero vector;
2) $H = \text{span}\{(1,3,1),(2,-5,-2)\}$ is a subspace;
3) $H = \ker T$ is a subspace, where $T(p) = p(0) - p(1) : P_4 \to \mathbf{R}$ is a linear transformation;
4) H is not a subspace because $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in H, but $A+B$ is not in H.

(5) 1) F; 2) T; 3) F; 4) T; 5) T; 6) F; 7) T; 8) F; 9) F; 10) F; 11) F; 12) T; 13) F; 14) F; 15) T.

Autumn 1999, Section 1

(1) Consider

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 2 \\ 2 & -2 & 0 & 2 & 2 \\ -1 & 1 & 2 & -3 & 1 \\ -2 & 2 & 1 & -3 & -1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

1) Write down the linear transformation given by the matrix
2) Is b_1 in the image of T? Is T onto? Is T one-to-one?
3) Solve the equation $Ax = b_2$ and express your solution in parametric vector form
4) Do columns of A span $\mathbf{R}^4$?
5) Find two linearly independent columns of A. Can you find three independent columns?

(2) 1) Compute the determinant of the matrix

$$A = \begin{bmatrix} x & a_1 & a_2 & a_3 \\ a_1 & x & a_2 & a_3 \\ a_1 & a_2 & x & a_3 \\ a_1 & a_2 & a_3 & x \end{bmatrix}$$

2) For what x is the matrix A invertible? For such x, find the determinant of $A^T A^{-2} A$
3) Make an educated guess on the determinant of

$$A_n = \begin{bmatrix} x & a_1 & \cdots & a_{n-1} \\ a_1 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & x \end{bmatrix}$$

(3) 1) Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

2) Use the result of the first part to solve the equation $A^2 x = (0,1,0,-1)$

3) Make an educated guess on the inverse of

$$A_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

and verify your guess

(4) 1) Find 3×3 matrices A and B, such that A and B are invertible but $A + B$ is not invertible

2) Find four linearly dependent vectors, such that any three are linearly independent

3) Find matrices A and B, such that $AB = O$ and $BA \neq O$. Is it possible to have $BA = I$? Explain

(5) True or False (*no reason needed*)

1) The row echelon form of a 5×3 matrix A has two pivots $\Rightarrow$ the solution of $Ax = 0$ has 1 free variable

2) For a 5×3 matrix A, the solution of $Ax = b$ can never be unique

3) For a 5×3 matrix A, $Ax = b$ does not have solution for some b

4) For a 5×3 matrix A, there is no 3×5 matrix B, such that $AB = I$

5) For a 5×3 matrix A, there is no 3×5 matrix B, such that $BA = I$

6) v_1, v_2, v_3 are independent $\Rightarrow$ no two vectors among v_1, v_2, v_3 are parallel

7) v_1, v_2, v_3, v_4 are independent $\Rightarrow$ Span$\{v_1, v_2, v_3\}$ is strictly smaller than Span$\{v_1, v_2, v_3, v_4\}$

8) Span$\{v_1, v_2, v_3\}$ is strictly smaller than Span$\{v_1, v_2, v_3, v_4\}$ $\Rightarrow$ v_1, v_2, v_3, v_4 are independent

9) v_1, v_2, v_3, v_4 are dependent $\Rightarrow$ Span$\{v_1, v_2, v_3\}$ is equal to Span$\{v_1, v_2, v_3, v_4\}$

10) Span$\{v_1, v_2, v_3\}$ is equal to Span$\{v_1, v_2, v_3, v_4\}$ $\Rightarrow$ v_1, v_2, v_3, v_4 are dependent

11) Columns of A are independent $\Rightarrow$ $Ax = b$ has solution for all b

12) Columns of A are independent $\Rightarrow$ $Ax = 0$ has only trivial solution

13) T is a linear transform, $T(v_1)$, $T(v_2)$, $T(v_3)$ are independent $\Rightarrow$ v_1, v_2, v_3 are independent

14) T is a linear transform, v_1, v_2, v_3 are independent $\Rightarrow$ $T(v_1)$, $T(v_2)$, $T(v_3)$ are independent

15) A is not invertible $\Rightarrow$ columns of A are independent

Answer to Midterm of Autumn 1999, Section 1

(1) The linear transformation given by the matrix is $T(x_1, x_2, x_3, x_4, x_5) = (x_1 - x_2 + x_3 + 2x_5, 2x_1 - 2x_2 + 2x_4 + 2x_5, -x_1 + x_2 + 2x_3 - 3x_4 + x_5, -2x_1 + 2x_2 + x_3 - 3x_4 - x_5)$.

Row operations reduce $[A \quad b_1 \quad b_2]$ to

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $Ax = b_1$ has no solution, and b_1 is not in the image of T. As a result, T is not onto. Moreover, T is not one-to-one because there are three free variables (alternatively, not one-to-one because $5 > 4$).

From the row operation, we also see that $Ax = b_2$ has solution $x_1 = 1 + x_2 - x_4 - x_5$ and $x_3 = 2 + x_4 - x_5$, with x_2, x_4, x_5 free. In vector form, the solution is $x = (1, 0, 2, 0, 0) + x_2(1, 1, 0, 0, 0) + x_4(-1, 0, 1, 1, 0) + x_5(-1, 0, -1, 0, 1)$.

Since there are two zero rows in the row echelon form of A (alternatively, since T is not onto), columns of A do not span $\mathbf{R}^4$?

The row echelon form of A has only two pivots, therefore there are at most two linearly independent columns of A. Any two non-parallel columns, such as the first and the third columns of A, are linearly independent.

(2) By subtracting third row from fourth, the second from third, and then the first from the second, we have

$$\det A = \det \begin{bmatrix} x & a_1 & a_2 & a_3 \\ a_1 - x & x - a_1 & 0 & 0 \\ 0 & a_2 - x & x - a_2 & 0 \\ 0 & 0 & a_3 - x & x - a_3 \end{bmatrix}$$

$$= (a_1 - x)(a_2 - x)(a_3 - x) \det \begin{bmatrix} x & a_1 & a_2 & a_3 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Then we add first column to second, the second to the third, and then the third to the fourth to obtain

$$\begin{aligned} \det A &= (a_1 - x)(a_2 - x)(a_3 - x) \det \begin{bmatrix} x & x+a_1 & x+a_1+a_2 & x+a_1+a_2+a_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= (a_1 - x)(a_2 - x)(a_3 - x)(-1)^{1+4}(x + a_1 + a_2 + a_3) \\ &= (x - a_1)(x - a_2)(x - a_3)(x + a_1 + a_2 + a_3) \end{aligned}$$

A invertible $\Leftrightarrow x \neq a_1, a_2, a_3$, and $-a_1 - a_2 - a_3$. For such x, $\det A^T A^{-2} A = \det A^T \det A^{-2} \det A = (\det A)(\det A)^{-2}(\det A) = 1$.

In general $\det A_n = (x - a_1) \cdots (x - a_{n-1})(x + a_1 + \cdots + a_{n-1})$.

(*Note: One can be quite sure that the formula for* $\det A_n$ *is the one above up to* $\pm$ *sign. In order to determine the sign, consider the special case* $a_1 = \cdots = a_{n-1} = 0$)

(3) Row operations reduce $[A \quad I]$ to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

The four columns on the right form A^{-1}. The solution of the equation $A^2 x = (0, 1, 0, -1)$ is $A^{-2}(0, 1, 0, -1) = (A^{-1})^2(0, 1, 0, -1) = (0, 1, 0, -1)$.

In general, we should have

$$A_n^{-1} = \begin{bmatrix} a & 1 & \cdots & 1 \\ 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1 \end{bmatrix}$$

From $A_n^{-1} A_n = A_n A_n^{-1} = I$, we get $a = 2 - n$.

(4) 1) $A = I_3$ and $B = -I_3$

2) $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, $v = (1, 1, 1)$. The four must be dependent because they are three dimensional.

3) Take $A = [1, \quad -1]$ (a 1×2 matrix) and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It is impossible to have $BA = I$ because this would imply $A = AI = ABA = OA = O$, which further implies $BA = BO = O$, a contradiction to $BA = I$.

(5)

1. T	2. F	3. T	4. T	5. F
6. T	7. T	8. F	9. F	10. T
11. F	12. T	13. T	14. F	15. F

Autumn 1999, Section 2

(1) Consider vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ -h \\ h \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 1 \\ h-2 \end{bmatrix}.$$

1) For what choice of h, is u in the span of v_1, v_2, v_3?

2) For what choice of h, is v_1, v_2, v_3 linearly independent?
3) For what choice of h, do v_1, v_2, v_3 span $\mathbf{R}^3$?
4) What is the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ such that $T(1,0,0) = v_1$, $T(0,1,0) = v_2 - v_3$, $T(0,0,1) = hv_1 + v_3$?

(2) Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 0 \\ -2 & 1 & 0 & -2 \\ 1 & 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 15 & 0.2 & 0.1 & 4 \\ \sqrt{\pi} & 2\pi & \pi & -\pi \\ 3 & \sqrt{12} & \sqrt{3} & -\sqrt{2} \\ \sqrt{2} & 2 & 1 & -\sqrt{2} \end{bmatrix}$$

1) Compute the determinants of A, B, AB, $2A$, A^3B^TA
2) Is A invertible? Is B invertible?
3) Are columns of A linearly independent? Are columns of AB linearly independent?

(3) Consider linear transformation

$$T(x_1, x_2, x_3) = (-x_1 + x_2 + x_3, x_1 - x_2 + x_3, x_1 + x_2 - x_3)$$

1) Show that T is invertible and find the inverse of T
2) Find a vector whose image under T is $(1, 3, 5)$. Is the vector unique?
3) Is T onto? Is T one-to-one?
4) Find the inverse of the composition $T^2 = TT$

(4) True or False (*no reason needed*)
1) More than 5 vectors in $\mathbf{R}^5$ are always linearly dependent
2) Less than 5 vectors in $\mathbf{R}^5$ are always linearly independent
3) n vectors in $\mathbf{R}^5$ are linearly dependent $\Rightarrow n > 5$
4) n vectors in $\mathbf{R}^5$ are linearly dependent $\Rightarrow n \leq 5$
5) v_1, v_2 are linearly independent $\Rightarrow v_1, v_2$ are not parallel
6) v_1, v_2 are linearly dependent $\Rightarrow v_1, v_2$ are parallel
7) $A_{m\times n}x = 0$ has only trivial solution $\Rightarrow$ columns of A span $\mathbf{R}^m$
8) $A_{m\times n}x = 0$ has only trivial solution $\Rightarrow$ columns of A are linearly independent
9) A linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^3$ can never be onto
10) A linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^3$ can never be one-to-one
11) $\det(cA^T) = |c| \det A$
12) A, B are square matrices and $AB = O \Rightarrow A$ is not invertible
13) A, B are square matrices and $AB = I \Rightarrow Ax = 0$ has only trivial solution
14) A row of A is all zero $\Rightarrow \det A = 0$
15) $\det A = 0 \Rightarrow$ a row of A is all zero

Answer to Midterm of Autumn 1999, Section 2

(1) Row operations reduce $[v_1 \quad v_2 \quad v_3 \quad v_4]$ to

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 2-h & 0 \\ 0 & 0 & (2-h)(h-1) & h-1 \end{bmatrix}$$

Thus
u in the span of v_1, v_2, $v_3 \Leftrightarrow h \neq 2$
v_1, v_2, v_3 linearly independent $\Leftrightarrow h \neq 1, 2$
v_1, v_2, v_3 span $\mathbf{R}^3 \Leftrightarrow h \neq 1, 2$
$T(x_1, x_2, x_3) = x_1T(1,0,0) + x_2T(0,1,0) + x_3T(0,0,1) = x_1v_1 + x_2(v_2 - v_3) + x_3(hv_1 + v_3) = (x_1 + 4x_2 + (h-2)x_3, x - 1 + (1+h)x_2, -x_1)$.

(2) By adding twice of first row third row to third, we have

$$\det A = \det \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 0 & 2 & -1 \end{bmatrix} = 5(-1)^{3+2} \det \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix} = -5 \det \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} = 30$$

By subtracting third column from second, the second column of B becomes zero. Thus $\det B = 0$.

$\det AB = \det A \det B = 0$.

$\det 2A = 2^4 \det A = 480$.

$\det A^3 B^T A = (\det A)^3 (\det B)(\det A) = 0$.

A is invertible? B is not invertible.

Columns of A are linearly independent because invertible. Columns of AB are linearly dependent because $\det AB = 0$.

(3) The matrix of T is

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Row operations reduce $[A \quad I]$ to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{bmatrix}$$

The three columns on the right form A^{-1}. Therefore

$$T^{-1}(x_1, x_2, x_3) = \frac{1}{2}(x_2 + x_3, x_1 + x_3, x_1 + x_2)$$

The vector $T^{-1}(1,3,5) = (4,3,2)$ has image $(1,3,5)$ under T. The vector is unique because T is invertible.

T is onto and one-to-one because it is invertible. The inverse of T^2 is given by the matrix

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}^2 = \frac{1}{4} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Therefore

$$T^{-2}(x_1, x_2, x_3) = \frac{1}{4}(2x_1 + x_2 + x_3, x_1 + 2x_2 + x_3, x_1 + x_2 + 2x_3)$$

(4)

1. T	2. F	3. F	4. F	5. T
6. T	7. F	8. T	9. F	10. T
11. F	12. F	13. T	14. T	15. F

Autumn 2000

(1) (25 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 2 & -1 & 1 & 0 & 1 \\ -1 & 3 & 2 & 5 & 1 \\ -1 & 2 & 1 & 3 & 0 \end{bmatrix}.$$

Answer the following and give full explaination.

1) Are the columns of A linearly independent?

2) Do the columns of A span $\mathbf{R}^4$?

3) Are the first, the third, the fourth, and the fifth columns linearly independent?

4) Is it possible to find three linearly independent columns of A?
5) Is it possible to find four columns of A to span $\mathbf{R}^4$?
6) Is the fourth column in the span of the first three columns?
7) Is the second column in the span of the first and the fourth columns?
8) Does $Ax = 0$ have nontrivial solutions?
9) Does $Ax = b$ have solutions for all $b \in \mathbf{R}^4$?

(2) (30 points) Consider vectors

$$u_1 = (0, 1, -1, 1), u_2 = (1, 0, 1, -1), u_3 = (-1, 1, 0, 1), u_4 = (1, -1, 1, 0).$$

1) Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be the linear transformation such that $T(e_1) = u_1$, $T(e_2) = u_2$, $T(e_3) = u_3$, $T(e_4) = u_4$. Compute the image of $(1, 2, 1, 2)$ under T.

2) Find the matrix of the linear transformation $S : \mathbf{R}^4 \to \mathbf{R}^4$ such that $S(u_1) = e_1$, $S(u_2) = e_2$, $S(u_3) = e_3$, $S(u_4) = e_4$.

3) Find a vector so that its image under S is $(1, 2, 1, 2)$? Is the vector unique? Explain.

4) Find the matrix of the linear transformation $R : \mathbf{R}^4 \to \mathbf{R}^4$ such that $R(u_1) = e_2$, $R(u_2) = e_1$, $R(u_3) = e_3$, $R(u_4) = e_4$. (*Hint: Find a linear transformation P such that $R = PS$*)

(3) (20 points) Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 1 & 1 & \sqrt{2} & \sqrt{3} \\ 1 & -1 & \sqrt{5} & \sqrt{6} \end{bmatrix}, \quad B = \begin{bmatrix} \pi & 1 & -\pi & -1 \\ 1 & -\pi & 1 & -\pi^{-1} \\ -\pi & 1 & -\pi^{-1} & 1 \\ -1 & -\pi^{-1} & 1 & \pi^{-1} \end{bmatrix}$$

1) Compute the determinants of A, B, A^T, B^2, ABA.
2) Is A invertible? Is B invertible? Explain.
3) Do columns of A span $\mathbf{R}^4$? Do rows of B span $\mathbf{R}^4$? Explain.

(4) (25 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 2 \\ -2 & 4 & 3 & -1 \\ -1 & 2 & 1 & -1 \end{bmatrix}.$$

1) Find one solution for each of the three systems of equations $Ax = e_1$, $Ax = e_2$, $Ax = e_3$.

2) Use your result in 1) to find one solution for each of the two systems of equations $Ax = (1, 2, 3)$, $Ax = (3, 2, 1)$.

3) Use your result in 1) to show that for any $b \in \mathbf{R}^3$, $Ax = b$ is consistent.

4) Use your result in 1) to find a matrix B, such that $AB = I_3$.

5) Suppose $A_{m\times n}x = b$ is consistent for all $b \in \mathbf{R}^m$. Explain why you can find $B_{n\times m}$, such that $AB = I_m$.

Answer to Midterm of Autumn 2000

(1) We have the row operations

$$A \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1) The columns of A are linearly dependent because the third and the fourth columns are not pivot.

2) The columns of A do not span $\mathbf{R}^4$ because the last row in the row echelon form consists of zeros.

3) The row operations on the first, the third, the fourth, and the fifth columns give us $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
Since we still have non-pivot column, the columns are linearly dependent.

4) If we chose the first, the third, and the fifth columns, the row operations give us $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, where all columns are pivot. Therefore the chosen columns are linearly independent.

5) The span of any four columns is no bigger than the span of all five columns. Since the all five columns do not span $\mathbf{R}^4$, any four columns cannot span $\mathbf{R}^4$.

6) The row operations on the first four columns give us $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since there is no row like $[0 \quad 0 \quad 0 \quad 0]$, the fourth column is in the span of the first three columns.

7) The row operations on the first, fourth, and the second columns give us $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since there is no row like $[0 \quad 0 \quad 0]$, the second column is in the span of the first and the fourth columns.

8) Since the columns of A are linearly dependent, $Ax = 0$ will have nontrivial solutions.

9) Since the columns of A do not span $\mathbf{R}^4$, $Ax = b$ has no solutions for some $b \in \mathbf{R}^4$.

(2) 1) $T(1, 2, 1, 2) = u_1 + 2u_2 + u_3 + 2u_4 = (3, 0, 3, 0)$.

2) S is the inverse of T. Thus its matrix is the inverse of $[T(e_1) \quad T(e_2) \quad T(e_3) \quad T(e_4)] = [u_1 \quad u_2 \quad u_3 \quad u_4]$. Using row operations, we find the inverse to be

$$\frac{1}{3}\begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 2 & 1 & -1 \\ -1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix}.$$

3) Since T and S are inverse to each other, we see $S(v) = (1, 2, 1, 2)$ is equivalent to $v = T(1, 2, 1, 2) = (3, 0, 3, 0)$. The invertibility of S also implies that this v is unique.

4) We have $R = PS$, where P is a linear transformation such that $P(e_1) = e_2$, $P(e_2) = e_1$, $P(e_3) = e_3$, $P(e_4) = e_4$. Therefore the matrix for R is the product of the matrix for P and the matrix for S, i.e., it is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 2 & 1 & -1 \\ -1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix}.$$

(3) Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 1 & 1 & \sqrt{2} & \sqrt{3} \\ 1 & -1 & \sqrt{5} & \sqrt{6} \end{bmatrix}, \quad B = \begin{bmatrix} \pi & 1 & -\pi & -1 \\ 1 & -\pi & 1 & -\pi^{-1} \\ -\pi & 1 & -\pi^{-1} & 1 \\ -1 & -\pi^{-1} & 1 & \pi^{-1} \end{bmatrix}$$

1) By exchanging the first and the third columns, and then exchanging the second and the fourth columns, we have

$$\det A = \det \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ \sqrt{2} & \sqrt{3} & 1 & 1 \\ \sqrt{5} & \sqrt{6} & 1 & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = 4.$$

By $\pi \times$ [4th row] + [1st row], the first row of B becomes zeros. Therefore $\det B = 0$. Then $\det(A^T) = \det A = 4$, $\det(B^2) = (\det B)^2 = 0$, $\det(ABA) = \det A \det B \det A = 0$.

2) $\det A \neq 0$ implies A is invertible. $\det B = 0$ implies B is not invertible.

3) Since the matrices are square, the invertibility is equivalent to columns spanning $\mathbf{R}^4$. Thus the columns of A span $\mathbf{R}^4$. The rows of B are columns of B^T. By $\det B^T = \det B = 0$, B^T is not invertible. Therefore the columns of B^T, i.e., the rows of B, do not span $\mathbf{R}^4$.

(4) 1) Row operations on $[A \quad e_1 \quad e_2 \quad e_3] = [A \quad I_3]$ give us

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -2 & 1 & -5 \\ 0 & 0 & 1 & 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

By taking the first five columns, we find a solution $v_1 = (-2, 0, -1, 1)$ of $Ax = e_1$. By taking the first four and the sixth columns, we find a solution $v_2 = (1, 0, 1, 0)$ of $Ax = e_2$. By taking the first four and the seventh columns, we find a solution $v_3 = (-5, 0, -3, 1)$ of $Ax = e_3$.

2) We have $A(v_1 + 2v_2 + 3v_3) = Av_1 + 2Av_2 + 3Av_3 = e_1 + 2e_2 + 3e_3 = (1, 2, 3)$. Therefore $u_1 = v_1 + 2v_2 + 3v_3 = (-15, 0, -8, 4)$ is a solution of $Ax = (1, 2, 3)$. Similarly, $u_2 = 3v_1 + 2v_2 + v_3 = (-9, 0, -4, 4)$ is a solution $Ax = (3, 2, 1)$.

3) For general $b = (b_1, b_2, b_3)$, the vector $b_1 v_1 + b_2 v_2 + b_3 v_3$ is a solution of $Ax = b$. Therefore $Ax = b$ is consistent for any b.

4) If we take $B = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} -2 & 1 & -5 \\ 0 & 0 & 0 \\ -1 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$, then we have $AB = [Av_1 \quad Av_2 \quad Av_3] = [e_1 \quad e_2 \quad e_3] = I_3$.

5) Since $A_{m\times n}x = b$ is consistent for all $b \in \mathbf{R}^m$, by taking $b = e_1, \cdots, e_n$, we find solutions $v_1, \cdots, v_n$ of the systems $Ax = e_1, \cdots, Ax = e_n$. In other words, $Av_1 = e_1, \cdots, Av_n = e_n$. Therefore if we take $B = [v_1 \quad \cdots \quad v_n]$, then we have $AB = [Av_1 \quad \cdots \quad Av_n] = [e_1 \quad \cdots \quad e_n] = I_m$.