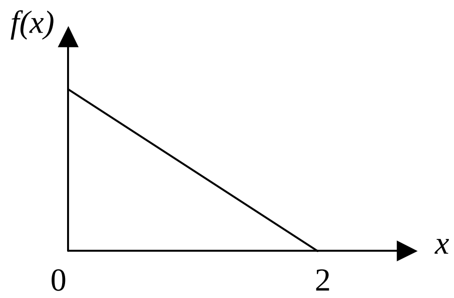
MATH 244 Applied Statistics

Quiz 3

Name ____________________ Student ID _______________ Tutorial section ____

Time allowed : 45 minutes

1. A continuous random variable X has a pdf shown in the following graph.



(a) Find $\Pr\left(X > \frac{1}{2}\right)$.

(b) Find $E(X)$ and $Var(X)$.

Solution

(a) Since the area under the pdf curve should be equal to 1, the y-intercept of the straight line should be 1. Hence the pdf is

$$f(x) = 1 - \frac{1}{2}x \quad , \ 0 < x < 2$$

$$\Pr\left(X > \frac{1}{2}\right) = \int_{1/2}^{2} \left(1 - \frac{1}{2}x\right) dx = \left[x - \frac{1}{4}x^2\right]_{1/2}^{2} = \frac{9}{16}$$

(b) $E(X) = \int_0^2 x\left(1 - \frac{1}{2}x\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^2 = \frac{2}{3}$

$$E(X^2) = \int_0^2 x^2\left(1 - \frac{1}{2}x\right) dx = \left[\frac{1}{3}x^3 - \frac{1}{8}x^4\right]_0^2 = \frac{2}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$$

2. Telephone calls enter a college switchboard according to a Poisson process on the average of two every three minutes.

   (a) What is the probability that at least 60 calls enter the switchboard within two hours?
   (b) What is the probability that the operator needs to wait longer than one hour to receive 30 calls?

   Solution

   (a) Let X be the number of calls enter the switchboard within two hours. Then $X \sim \wp(80)$. Using normal approximation to Poisson,

$$\Pr(X \geq 60) = \Pr\left(\frac{X-80}{\sqrt{80}} \geq \frac{59.5-80}{\sqrt{80}}\right) \approx 1 - \Phi(-2.29) = 0.9890$$

   (b) Let W be the waiting time until the 30th call. Then

$$W \sim \Gamma\left(30, \frac{2}{3}\right) \Rightarrow \frac{4}{3}W \sim \Gamma\left(30, \frac{1}{2}\right) \equiv \chi^2_{60}$$

   Hence $$\Pr(W > 60) = \Pr\left(\frac{4}{3}W > 80\right) \approx 0.05$$ (From Chi-square table.)

3. From past experience an instructor knows that the test score of a student taking his final examination is a random variable with mean 71 and standard deviation 19. Assume that the score is normally distributed.

   (a) What is the probability that a student can get a score larger than 75?
   (b) What is the probability that the average score of the students in a class of size 60 exceeds 75?
   (c) What passing mark should he set such that 90% of the students in his class will pass the examination?
   (d) If the normal assumption is violated (i.e. the actual distribution of the score is not normal), will the calculations in (a), (b) and (c) still be valid? Why?

Solution

(a) $\Pr(X > 75) = 1 - \Phi\left(\frac{75-71}{19}\right) = 1 - \Phi(0.21) = 1 - 0.5832 = 0.4168$

(b) $\Pr(\overline{X} > 75) = 1 - \Phi\left(\frac{75-71}{19/\sqrt{60}}\right) = 1 - \Phi(1.63) = 1 - 0.9484 = 0.0516$

(c) Let c be the passing mark he should set. Then

$$\Pr(X \geq c) = 0.9 \Rightarrow 1 - \Phi\left(\frac{c-71}{19}\right) = 0.9 \Rightarrow \frac{c-71}{19} = -1.282 \Rightarrow c = 46.642$$

(d) If the normal assumption is violated, the calculations in (a) and (c) will be incorrect. However, the calculation in (b) is still valid because by central limit theorem, the sample mean of a random sample will follow a normal distribution approximately when the sample size is large.

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