MATH246

FALL SEMESTER 1997

FINAL EXAMINATION

2. A discrete-time random process X_n is defined as follows. A coin is tossed where $P(\text{head}) = \frac{2}{3}, P(\text{Tail}) = \frac{1}{3}$. If the outcome is head, $X_n = (-1)^n$ for all n; if the outcome is tail, $X_n = (-1)^{n+1}$ for all n.

   (5pts) a. Find the pmf for X_n.

   (5pts) b. Find the joint pmf for X_n and X_{n+k}.

   (5pts) c. Find the mean and autocovariance functions of X_n.

3. Let $Y(t) = 2X(t) - 3X(t+1) + 4X(t+2)$, where $X(t)$ is a Gaussian random process.

   (8pts) a. Find the mean and autocovariance of $Y(t)$.

   (7pts) b. Find the pdf of $Y(t)$.

4. Let $Y(t) = X(t) - 2X(t+1) + 3X(t+2)$, where $X(t)$ is a wide-sense stationary random process.

   (8pts) a. Determine whether $Y(t)$ is also a wide-sense stationary random process.

   (7pts) b. Find the pdf of $Y(t)$ if $X(t)$ is also a Gaussian random process.

5. Let $D_n = X_n - 2X_{n-1}$, where X_n is a zero-mean, WSS random process.

   (8pts) a. Find $R_D(n, n+k) = R_D(k)$ and $S_D(f)$ in terms of $R_X(k)$ and $S_X(f)$.

   (7pts) b. Find $E[D_n^2]$.

6a. A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix:
(7pts)

$$P = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} .3 & .2 & .5 \\ .5 & .1 & .4 \\ .5 & .2 & .3 \end{bmatrix} \end{array}$$

and initial distribution $p_0 = .5$ and $p_1 = .5$. Determine the probabilities $\Pr\{X_2 = 0\}$ and $\Pr\{X_3 = 0\}$.

b. Consider the Markov chain whose transition probability matrix is given by
(8pts)

$$P = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} .4 & .3 & .2 & .1 \\ .1 & .4 & .3 & .2 \\ .3 & .2 & .1 & .4 \\ .2 & .1 & .4 & .3 \end{bmatrix} \end{array}.$$

Suppose that the initial distribution is $p_i = P\{X_0 = i\} = \frac{1}{4}$ for $i = 0, 1, 2, 3$. Show that $\Pr\{X_n = k\} = \frac{1}{4}$, $k = 0, 1, 2, 3$, for all n.

7. Let $X_1, X_2, \ldots$ be i.i.d. discrete random variables with probability distribution $\{p_k; k = 0, 1, 2, \ldots\}$.

i.e.,

$$P\{X_i = k\} = p_k, \qquad k = 0, 1, 2, \ldots$$

Let

$$S_0 \equiv 0$$

$$S_n = X_1 + \cdot + X_n \qquad n \geq 1$$

We already know that $\{S_n\}$ is a Markov chain.

(8pts) a. Find the one-step transition probabilities $P\{X_{n+1} = j | X_n = i\}$ in terms of the $p_k^{\prime} s$.

(7pts) b) Write down the one-step transition probability matrix P.