COMP 272 Theory of Computation

April 13, 2000, **Midterm Examination**

Name: ______________________________ E-mail: __________________________

Student No: __________________________ Tutorial: __________________________

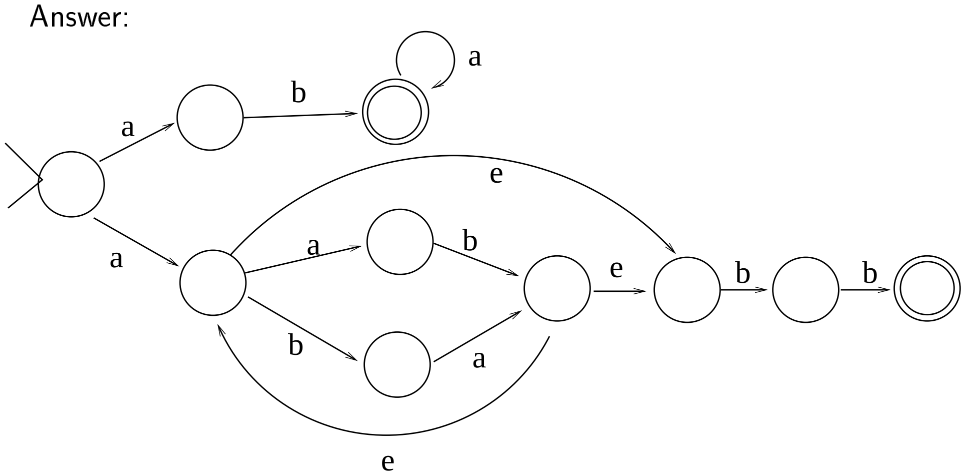
You have 2 and a half hours to solve 6 problems for a total of 100 points. Answer all problems in the space provided. You may also use the back of the test papers. Good luck!

1) (15 points)

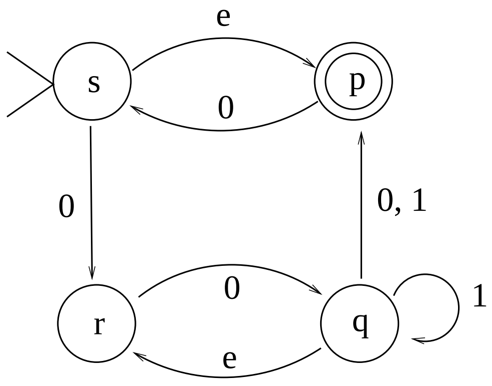
a) (5 points) For the following regular expression over $\{a, b\}$, draw an NFA that accepts the corresponding language:

$$aba^* \cup a(ab \cup ba)^*bb$$

Answer:



b) (10 points) Convert the following nondeterminstic FA to a deterministic FA:



Answer:

$E(s) = \{s, p\}$
$E(q) = \{q, r\}$
$E(r) = \{r\}$
$E(p) = \{p\}$

	0	1
$\{s, p\}$	$\{r, s, p\}$	$\emptyset$
$\{r, s, p\}$	$\{r, s, p, q\}$	$\emptyset$
$\{r, s, p, q\}$	$\{r, s, p, q\}$	$\{r, p, q\}$
$\{r, p, q\}$	$\{r, s, p, q\}$	$\{r, p, q\}$
$\emptyset$	$\emptyset$	$\emptyset$

2) (20 points)

a) (10 points) Find a CFG for the following language

$$\{a^i b^j c^k | 0 \leq i \leq j + k\}$$

Answer:

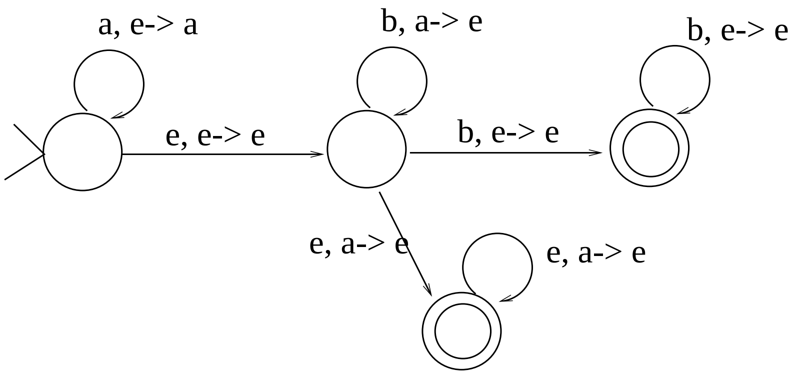
$G = (V, \Sigma, S, R)$ where

$V = \{S, X, a, b, c\}$
$\Sigma = \{a, b, c\}$
$R = \{\ S \rightarrow aSc,$
$S \rightarrow Sc,$
$S \rightarrow X,$
$X \rightarrow aXb,$
$X \rightarrow Xb,$
$X \rightarrow e\ \}$

b) (10 points) Construct a pushdown automaton that accepts the following language

$$\{a^m b^n | m \neq n, m, n \geq 0\}.$$

It is sufficient to draw a state diagram.



3) (15 points)

a) (3 points) State the Pumping Theorem for regular languages.

Answer:

Let L be a regular language. There is an integer $n \geq 1$ such that any string $w \in L$ with $|w| \geq n$ can be rewritten as $w = xyz$ such that $y \neq e$, $|xy| \leq n$, and $xy^i z \in L$ for each $i \geq 0$.

b) (12 points) Using the Pumping Theorem stated in (a), show that the following language is not regular:

$$L = \{a^i b^j c^k : i, j, k \geq 0 \text{ and } i + j < k\}$$

Answer:

Suppose to the contrary that L is regular. Let n be the integer in the pumping theorem. Consider the string $w = a^n b^n c^{2n+1}$ which is in L and has length longer than n. Let x, y, z be as specified in the pumping theorem. Then $x = a^p$, $y = a^q$, and $z = a^r b^n c^{2n+1}$, where $q > 0$, and $p + q + r = n$. Consider xy^2z; the number of a and b in $xy^2z = 2n + q \geq 2n + 1 =$ number of c. Thus, $xy^2z \notin L$, contradicting to the theorem.

4) (20 points)
For each of the statement below, decide whether it is true or false. If it is true, prove it; if not, give a counterexample.

a) (6 points) If L_1 is nonregular and L_2 is regular, then $L_1 \cup L_2$ is nonregular.

Answer:

False. Let $L_1 = \{a^n b^n : n \geq 0\}$ and $L_2 = a^* b^*$. Then $L_1 \cup L_2 = L_2$ which is regular.

b) (7 points) Let C be a context-free language, and R be a regular language over Σ. Define the following language

$$L_{C,R} = \{w : w \in C \text{ and there exists } x \in R \text{ and } u \in \Sigma^* \text{ such that } w = ux\}.$$

Then $L_{C,R}$ is always context-free.

Answer:

True. Note that $L_{C,R} = C \cap \Sigma^* R$. Since the set of regular languages is closed under Kleene star and concatenation, $\Sigma^* R$ is regular. Since intersection of a context-free language and a regular language gives a context-free language, $L_{C,R}$ is context-free.

c) (7 points) If L is context-free, then $\text{mirror}(L) = \{ww^R : w \in L\}$ is context-free.

Answer:

False. Let $L = \{a^n b^n : n \geq 0\}$. Then $\text{mirror}(L) = \{a^n b^{2n} a^n : n \geq 0\}$ which is not context-free.

5) (15 points)
A CFG is called *right-linear* if every grammar rule takes one of the forms $B \to xC$, $B \to x$, or $B \to e$, where B and C are nonterminals and $x \in \Sigma^*$. Prove that if a CFG G is right-linear, $L(G)$ is regular.

Answer:

First, convert each rule of the form $B \to \sigma_1\sigma_2 \cdots \sigma_n C$, $n > 1$ into several rules by introducing dummy nonterminals: $B \to \sigma_1 X_1$, $X_1 \to \sigma_1 X_2, \cdots, X_{n-1} \to \sigma_n C$. Also, convert each rule of the form $B \to \sigma_1\sigma_2 \cdots \sigma_n$, $n > 1$ into several rules: $B \to \sigma_1 X_1$, $X_1 \to \sigma_1 X_2, \cdots, X_{n-1} \to \sigma_n$.

Then, construct a NFA M as follows:

- Introduce a state corresponds to each nonterminal, and introduce one final state f.
- For each rule of the form $X \to \sigma Y$, $\sigma \in \Sigma$, create a link from state X to state Y and label it σ.
- For each rule of the form $X \to \sigma$, $\sigma \in \Sigma$, or $X \to e$, create a link from state X to the final state f and label it σ or e, respectively.

You can prove that $L(G) = L(M)$ by induction on the length of a derivation/computation.

6) (15 points)
Prove that the following language is not regular.

$$L = \{a^i b^j c^k : i, j, k \geq 0, i \neq j \text{ or } j \neq k\}$$

Answer:

Method 1

Suppose L is regular. Note that $\bar{L} \cap a^*b^*c^* = \{a^n b^n c^n : n \geq 0\}$.

Since regular languages is closed under complement and intersection, $\bar{L} \cap a^*b^*c^*$ produces a regular language. However, $\{a^n b^n c^n : n \geq 0\}$ is known to be not regular. Contradiction.

Method 2

Suppose to the contrary that L is regular. Let n be the integer in the pumping theorem. Consider the string $w = a^n b^{n+n!} c^{n+n!}$ which is in L and has length longer than n. Let x, y, z be as specified in the pumping theorem. Then $x = a^p$, $y = a^q$, and $z = a^r b^{n+n!} c^{n+n!}$, where $q > 0$, and $p + q + r = n$. Consider $xy^{n!/q}z$ (since $0 < q \leq n$, $n!/q \in \mathcal{Z}$), then the number of a in that string $= n + q(n!/q) = n + n!$ which is equal to the number of b's and the number of cs. Thus, there exists an i such that $xy^i z \notin L$, contradicting to the theorem.