COMP 272 Midterm: Suggested answers

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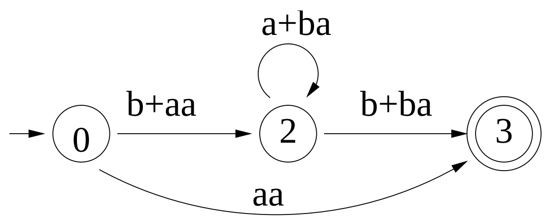
Problem 1

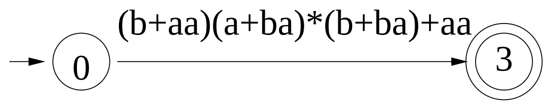
Same as the proof that $2^{\mathbf{N}}$ is uncountable in the lecture notes except that a string a^n now plays the role of the number n.

Problem 2

a. After eliminating state 1:



b. The regular expression is $(b + aa)(a + ba)^*(b + ba) + aa$.



Problem 3

We will prove that the following property denoted by $\mathcal{P}(n)$ is true for all $n \in \mathbf{N}$:

Property 1 *For all $w \in \{a, b\}^n$, let $(0, w) \vdash^n (q, \lambda)$.*

- *If $q = 0$ then $w = \lambda$.*
- *If $q = 1$ then $w \in L$ and w ends with a.*
- *If $q = 2$ then $w \in L$ and w ends with b.*
- *If $q = 3$ then w has one more occurence of ab than ba and ends with b.*

- *If $q = 4$ then w has one more occurence of ba than ab and ends with a.*

We prove by induction on n that $\mathcal{P}(n)$ is true for all integer n. Obviously $\mathcal{P}(0)$ is true.

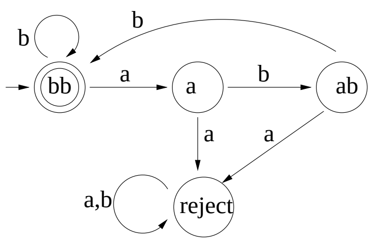
Now we assume that for some integer n, $\mathcal{P}(n)$ is true and we prove that $\mathcal{P}(n+1)$ is true too. *This part is fairly trivial and is left to the reader.*

We have just proven that $\mathcal{P}(n)$ is valid for any integer n, as a corollary it yields that the automaton is correct.

Comments This proof technique that I call the invariants method makes it relatively easy to prove an automata. Remember that there should be one invariant per state, and they must exclude each other (in other words, they should precisely describe each state). In our case, the invariants can be simplified if you first notice that L is the set of words that start and end with the same letter, but this requires a proof!

There can always be more clever ways than the invariant method that make the proof shorter and nicer, however you have to be very careful. I would just mention an attempt of a student to use state elimination, I like the idea, however there was a serious problem in it.

Problem 4



a.

b. $(b + abb)^*$ or the equivalent $(b^*(abb)^*)^*$

Problem 5

a. See the lecture notes.

b. We will prove that L is not regular by contradiction. So we assume that L is regular. Let N be the pumping constant for L. Let $w = ab^N a^N$. By the pumping lemma, there are three strings x, y, z such that $w = xyz$, $|xy| \leq N$, $|y| \geq 1$ and for all $i \geq 0$, $xy^i z \in L$.

If $x \neq \lambda$ then $y = a^k$ for some $1 \leq k \leq N$ and $z = b^{N-k}a^N$. So the pumping lemma yields $xy^0 z = xz = ab^{N-k}a^N \in L$, which is obviously wrong so we reached a contradiction. On the other hand, if $x = \lambda$ then we observe that the string $xy^0 z = xz$ starts with a b, so we have a contradiction too.

Comments You should read the statement of the pumping lemma carefully. In particular, N is fixed (actually you can make it larger, but I see no point in it) so you are not supposed to chose it. Then you can chose any word w you want. However, once w is fixed you cannot chose x, y and z. In short, you can only chose w and i, the rest is fixed.

Problem 6

a. A state of the new machine is labelled by a pair of states $(q_1, q_2) \in Q_1 \times Q_2$. There is a transition between states (q_1, q_2) and (q_1', q_2') if and only if there is a transition between q_1 and q_1' and between q_2 and q_2' with the same letter. A state (q_1, q_2) is final if and only if both q_1 and q_2 are final.

b. $Q = Q_1 \times Q_2$

$s = (s_1, s_2)$

$F = F_1 \times F_2$

$\delta = \{((q_1, q_2), a, (q_1', q_2')) : (q_1, a, q_1') \in \delta_1 \text{ and } (q_2, a, q_2') \in \delta_2\}$